GSTS Problem Session

Week 4

1. Let $G = \langle x, y \mid x^3 = y^2 \rangle$. Show that the map $f : G \to B_3$ generated by $f(x) = \sigma_1\sigma_2$ and $f(y) = \sigma_1\sigma_2\sigma_1$ is well-defined and an isomorphism.

2. Define the length homomorphism $\ell : B_n \to \mathbb{Z}$ as follows: if w is a word in the σ_i (and their inverses), let $\ell(w)$ be the sum of the exponents of the σ_i in w (e.g. $\ell(\sigma_1\sigma_2^2\sigma_3^{-4}) = 1 + 2 - 4 = -1$).

   (a) Show that ℓ is well-defined.

   (b) Recall from last week that we showed the abelianization of B_n is isomorphic to $\mathbb{Z}$ for $n \geq 2$. Show that the map ℓ is the same as the abelianization map $B_n \to \mathbb{Z}$.

3. Show that all the σ_i are conjugate to each other. Draw a braid demonstrating this conjugacy.

4. Let $\delta = \sigma_1\sigma_2\sigma_1$ and $\gamma = \sigma_2\sigma_1^{-1}$.

   (a) Show that $\delta\gamma\delta^{-1} = \gamma^{-1}$. In other words, γ is conjugate to its inverse.

   (b) Show that if β is any braid conjugate to itself, then $\ell(\beta) = 0$.

5. (a) Is PB_n normal in B_n?

   (b) Calculate the index of PB_n in B_n.

   (c) Describe the subgroup $PB_2 \leq B_2 \cong \mathbb{Z}$.

   (d) Show that the center of B_n is contained in PB_n without using the fact that $Z(B_n)$ is generated by the "full-twist" braid.

Note: These exercises were taken from Office Hour 18 of Office Hours with a Geometric Group Theorist.