Groups

- Given a grp G w/ finite gen. set S,
  - get Cayley graph $\Gamma(G,S)$ which gives a geometric (metric) "picture" of G

Q: - What happens at ∞?
- As we approach ∞ do we get a space w/ a certain structure?
- G acts on $\Gamma(G,S)$ in a natural way
  - does this give an action on the space @ ∞?

(probably won't have time to get to these q's in this talk – will begin with something more elementary – counting) but no less important

Recall Def: A map $f: X \to Y$ is a q-isom embedding if $\exists\, K \geq 1$ and $C \geq 0$ s.t.

$$\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$$

$\forall\, x_1, x_2 \in X$.

Def: A q-isom embedding f is a q-isom if $\exists$ some $D \geq 0$ s.t. $\forall\, y \in Y$ $\exists$ some $x \in X$ s.t. $d_Y(f(x), y) \leq D$.

(A q. motivating our counting)

Motivating Question: Are $[0,\infty)$ and $\mathbb{R}$ quasi-isometric?
- Quasi-isometry preserves coarse geometry of a metric space
- Heuristically, one feature this coarse geometry should preserve might be # of "directions" in which a metric space extends $\rightsquigarrow$ ends

Ex

$\mathbb{R}$ extends to ∞ in two directions

$[0,\infty)$ extends to ∞ in one direction

Prop: $[0,\infty)$ and $\mathbb{R}$ are not quasi-isomorphic

Pf: Let $f: \mathbb{R} \to [0,\infty)$ be a quasi-isometry.

As $n \to \infty$, $f(n) \to \infty$ and $f(-n) \to \infty$ ♦ (coarse surj cond.)

$\exists$ const. C s.t. $|f(n+1) - f(n)| \leq C$ $\forall\, n \in \mathbb{R}$. (from f being a q-isom embedding)

Fix $L \in [0,\infty)$.

Let x be the largest positive integer s.t. $f(x) < L$ (possible by ♦)

$\Rightarrow f(x+1) \geq L$ so $f(x) \in [L-C, L+C]$

Similarly let y be the largest negative integer s.t. $f(y) < L$

$\Rightarrow f(y) \in [L-C, L+C]$

So $|f(x) - f(y)| \leq 2C$

Then $\exists$ const M s.t. $x \leq x - y = |x-y| \leq M$ (f q-isom embedding)

By choosing L large, get $\rightarrow\leftarrow$ (by ♦)

ex. where different # of "ends" corresponds to not quasi-isometric

Want: To do Geometric Group Theory

- Will restrict attention to graphs b/c the metric spaces we want to consider are Cayley graphs (and with that we come to our first def)

Def: Γ a connected, locally finite graph, $B(n)$ the ball of radius n about a fixed vertex $v \in V(\Gamma)$. The number of ends of Γ is

$$e(\Gamma) = \lim_{n\to\infty} \|\Gamma \setminus B(n)\|$$

(exists since graphs have finite # of components outside any subset)

Def (preliminary): G a grp w/ finite gen. set S. The number of ends of G is $e(\Gamma(G,S))$.

- NTS: def. does not depend on choice of gen set S

Lemma: G, H q-isom grps w/ finite gen sets R, S resp. Then $e(\Gamma(G,R)) = e(\Gamma(H,S))$.

~~Notation~~/ ~~$N(x,r) = \{y \in$~~ Def:

(M,d) metric space. For $r \geq 0$, the closed r-nbhd of a pt $x \in M$ is denoted $N(x,r) = \{y \in M \mid d(x,y) \leq r\}$

and the nbhd of a subset $Q \subset M$ is

$$N(Q,r) = \bigcup_{x \in Q} N(x,r).$$

$f: G \to H$ q-isom

PF: $\forall g_1, g_2 \in G$ w/ $d_R(g_1,g_2) = 1$, $d_S(f(g_1), f(g_2)) \leq K + C$ (f q-isom embedding) * (cons 4)

- Let $F = d_S(f(1), 1_H)$
- Choose $E > K(2C + K + F)$
- WLOG assume $C \geq 1$
- fix $n \in \mathbb{R}$
- Let $g, h \in \Gamma(G,R) \setminus N_R(1_R, En + E)$ s.t. g, h is same connected component
  - Have path $g_0 = g \to g_1 \to g_2 \to \cdots g_k = h$ s.t. $d_R(g_i, 1) > En + E$
  - So $d_S(f(g_i), 1_S) \geq d_S(f(g_i), f(1_R)) - F$ (triangle ineq)

$$\geq \frac{1}{K} d_R(g_i, 1_R) - C - F \quad \text{(f q-isom embed)}$$

$$\geq \frac{1}{K}(En + E) - C - F$$

$$> n + C + F \quad \text{(tracing through ineq's + simp)}$$

//

- Construct path from $f(g)$ to $f(h)$ in $\Gamma(H,S)$
  - edge $g_i g_{i+1}$ has length 1 in $\Gamma(G,R)$
    so $\exists$ path of length at most $K+C$ btw $f(g_i)$ + $f(g_{i+1})$ * (obs 2)
  - connect these paths
- Since $d_S(f(g_i), 1_H) > n + C + F$ for each i,
  a path in $\Gamma(H,S)$ from $f(g)$ to $f(h)$
  cannot intersect $N_S(n, 1_H)$
  $\Rightarrow$ for any path $g \rightsquigarrow h$ in $\Gamma(G,R) \setminus N_R(1_G, En+E)$
  can find corresponding path from $f(g)$ to $f(h)$ in $\Gamma(H,S)\setminus N_S(1_H, n)$
- Correspondence gives map taking each unbounded connected component in $\Gamma(G,R)\setminus N_R(1_G, En+E)$
  to an unbounded connected subgraph of some unbounded connected component in $\Gamma(H,S)\setminus N_S(1_H, n)$

$\Rightarrow \|\Gamma(G,R)\setminus N_R(1, En+E)\| \geq \|\Gamma(H,S)\setminus N_S(1_H, n)\|$

taking $n \to \infty$ gives $e(\Gamma(G,S)) \geq e(\Gamma(H,S))$

(Repeating arg w/ G+H swapped + using quasi-inverse of f completes proof) - note $\Gamma(G,S) \sim \Gamma(G,R)$ so def of ends well def.

Also $\Rightarrow$ quasi-isom grps have same # of ends

Thm (Freudenthal-Hopf) Every finitely generated grp has either zero, one, two, or infinitely many ends.

Pf: G a grp w/ finite gen set S. \$ $e(G) = k \geq 3$.
- Note: G has 0 ends iff G finite $\Rightarrow$ G infinite
- So $\exists\, n \in \mathbb{N}$ s.t. $\Gamma(G,S)\setminus B(n)$ has k unbounded connected components
  $\exists\, g \in G \quad n < d(e,g) < 2n$
  $\exists\, g \in G$ sends $B(n)$ sends $B(n)$ at least n from origin
  $g \cdot B(n)$ sends $B(n)$ sends
  ~~$d(g,1) \leq d(ga,a) \leq d(ag,1) + d(a,1)$~~

- $\exists\, g \in G$ w/ $d_S(1,g) > 2r$
  - $N(1,r)\cdot g \cap N(1,r) = \emptyset$ since $ag \in \cap \Rightarrow d_S(g,1) \leq d_S(ag,a) \leq d_S(ag,1) + d_S(a,1) \leq 2r$

$\Rightarrow N(1,r)\cdot g$ contained in some unbounded component of $\Gamma(G,S)\setminus N(1,r)$
  - and divides it into at least $k-1$ unbounded connected components

- Let $C = N(1,r)\cdot g \cup N(1,r)$. Then $\|\Gamma(G,S)\setminus C\| \geq 2k-2 > k \rightarrow\leftarrow$.

Thm: G a finitely generated grp w/ a finite index subgrp N. Then $e(G) = e(N)$ $*$

~~Recall~~:

Thm: If G and H are two fin. gen grps and G acts on H. ✦
Then $e(G \rtimes H) = 1$

Recall: B_n

 braids on n-strings
group operation: stacking

Have surjective homomorphism $\varphi : B_n \to S_n$

Ex. $B_3 \to S_3$

 $\mapsto (12)$

$\ker \varphi = PB_n$ normal subgrp of index $n!$
$\Rightarrow e(B_n) = e(PB_n)$ by Thm $*$

How does this help?

Can use combing to prove

$PB_n \cong F_{n-1} \rtimes PB_{n-1}$

Idea: $\exists$ a surj hom from $PB_n \to PB_{n-1}$
obtained by removing the nth string

■ ker of this map consists of all pure braids for which only the nth string is braided through other strings

▲ combing $\Rightarrow$ ker is free grp F_{n-1}

■ Since $\exists$ an inj hom $PB_{n-1} \to PB_n$
adding nth string w/o additional braiding,
can view PB_{n-1} as a subgrp of PB_n

▲ PB_{n-1} acts on F_{n-1} by conj

$\Rightarrow PB_n \cong F_{n-1} \rtimes PB_{n-1}$

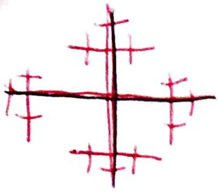
If $n \geq 3$, PB_{n-1} infinite

so by ✦ $e(PB_n) = 1$ $\Rightarrow e(B_n) = 1$

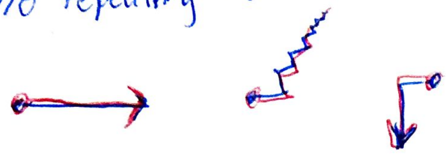
$B_2 \cong \mathbb{Z}$, $B_1 \cong \{e\}$

so $e(B_2) = 2$ and $e(B_1) = 0$

Ends of $\Gamma(F^2, \langle a,b\rangle) =: T$



Def: A *ray* in T starts at some base vertex + continues indef. w/o repeating vertices



Define an equiv. rel. $r_1 \sim r_2$ if $r_1 \cap r_2$ is a ray.

Denote eq. class $[r]$

ends of T are these equiv. classes

set of equiv. classes denoted ∂T

Exercise: Given $e \in \partial T$ and $v \in T$, $\exists!$ ray in T ~~representing~~ that represents e and is based at v.

// $F_2 \curvearrowright \partial T$

Consider

gens

Have ray r_1: $id \to b \to b^2 \to b^3 \to \dots$

$a \cdot [b^\infty] = [ab^\infty]$ "a moves all ends above or below axis, step to right"

Remains to consider action of a on

$id \to a \to a^2 \to a^3 \to \dots$

$a \cdot [a^\infty] = [a^\infty]$, $[a^{-\infty}]$ also fixed by a

// Notion of distance btwn ~~rays~~ ends

e, ε ends, v base pt

r_e, r_ε ! rays representing ends

s final vertex where r_e, r_ε overlap

$$d_\infty(e,\varepsilon) := 2^{-d(v,s)}$$

Exercise: Prove ∂T is a totally disconnected, perfect compact metric space, i.e. is homeo to the Cantor set

$C = C'$ all lim pts of C

"Rmk: Ends of spaces + grps turn out to be a cohomological invariant"