Second (co)homology groups

Let G be a group. The purpose of this sheet is to prove the Hopf formula for $H_2(G,\mathbb{Z})$ and to give a description of $H^2(G,A)$ in terms of group extensions. But before that let me recall some preliminaries about resolutions.

Preliminaries

Let $P_n := \mathbb{Z}[G]^{\otimes(n+1)}$ be a G-module equipped with the diagonal G-action. Define the $\mathbb{Z}[G]$-linear homomorphism $d_n\colon P_n \to P_{n+1}$ by the rule

$$d_n(g_0 \otimes g_1 \otimes \ldots \otimes g_n) = \sum_{i=0}^{n+1} (-1)^i g_0 \otimes \ldots \otimes g_{i-1} \otimes \widehat{g_i} \otimes g_{i+1} \otimes \ldots \otimes g_{n+1}.$$

One can check that $d_n \circ d_{n-1} = 0$ and that the sequence

$$\ldots \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \to \ldots$$

is a projective resolution of the trivial G-module $\mathbb{Z}$.

Let $Q_n := \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathbb{Z}[G]^{\otimes n}$ be the free $\mathbb{Z}[G]$-module generated by the set G^n. Let denote an element of the given basis of Q_n by a symbol $[g_1|g_2|\ldots|g_n]$. Define the $\mathbb{Z}[G]$-linear homomorphism $d'_n\colon Q_n \to Q_{n-1}$ by the rule:

$$d'_n[g_1|g_2|\ldots|g_n] = g_1[g_2|\ldots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\ldots|g_{i-1}|g_i g_{i+1}|\ldots|g_n] + (-1)^{n+1}[g_1|\ldots|g_{n-1}].$$

Again, one can check that $d'_n \circ d'_{n-1} = 0$ and that the sequence

$$\ldots \xrightarrow{d'_{n+1}} Q_n \xrightarrow{d'_n} Q_{n-1} \xrightarrow{d'_{n-1}} Q_{n-2} \to \ldots$$

is a projective resolution of the trivial G-module $\mathbb{Z}$.

Problem 1. Construct a $\mathbb{Z}[G]$-linear isomorphism between resolutions $(P_\bullet, d_\bullet)$ and $(Q_\bullet, d'_\bullet)$.

Now define the subcomplex $D_\bullet \subset Q_\bullet$. Let D_n be the free $\mathbb{Z}[G]$-module generated by the elements $[g_1|g_2|\ldots|g_n]$ such that $g_i = e \in G$ for some i.

Problem 2. Show that $d'_n(D_n) \subset D_{n-1}$. Show that the quotient complex $Q_\bullet/D_\bullet$ is a projective resolution of the trivial G-module $\mathbb{Z}$. This quotient complex is known as **normalized bar complex.**

Definition 1. Let A be a G-module. The **n-th homology group** $H_n(G,A)$ of the group G with coefficient in A is $H_n(P_\bullet \otimes_{\mathbb{Z}[G]} A) = H_n(Q_\bullet \otimes_{\mathbb{Z}[G]} A)$. The **$n$-th cohomology group** $H^n(G,A)$ of the group G with coefficient in A is $H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(P_\bullet, A)) = H^n(\mathrm{Hom}_{\mathbb{Z}[G]}(Q_\bullet, A))$.

Problem 3. Show that $H^*(G,A)$ can be computed using the following chain complex (C^n, δ^n). Here the group $C^n(G,A)$ is the abelian group of all functions $\varphi\colon G^n \to A$ such that $\varphi(g_1,\ldots,g_n) = 0$ if some $g_i = e \in G$. Define the differentials $\delta^n\colon C^{n-1}(G,A) \to C^n(G,A)$ by the rule:

$$(\delta^n\varphi)(g_1,g_2,\ldots,g_n) = g_1\varphi(g_2,\ldots,g_n) + \sum_{i=1}^{n-1}(-1)^i\varphi(g_1,\ldots,g_{i-1},g_i g_{i+1},\ldots,g_n) + (-1)^{n+1}\varphi(g_1,\ldots,g_{n-1}).$$

Group Extensions

This section is devoted to give more down-to-earth description of $H^2(G, A)$ in terms of group extensions.

Definition 2. Let A be an abelian group and let G be a group. We say that a third group E is **an extension** of G by A if there exists the short exact sequence:

$$0 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1.$$

This means that $i(A)$ is a normal subgroup of E and the kernel of p is precisely the group $i(A)$.

Problem 4. Consider the E-action on A by conjugation. Show that this action is actually a G-action (i.e. $A \subset E$ acts trivially on A).

Definition 3. We say that extensions E and E' of the group G by A are equivalent if there exists a group isomorphism $\beta\colon E \to E'$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & E & \xrightarrow{p} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & A & \xrightarrow{i'} & E' & \xrightarrow{p'} & G & \longrightarrow & 1 \end{array}$$

Problem 5. Show that two equivalent extensions give the same G-action on A.

So now we fix a G-action on A and try to describe all group extension of G by A with the given G-action up to equivalence.

We say a section $\sigma\colon G \to E$ of p, $p\sigma = \mathrm{Id}_G$, is **based** if it preserves the neutral element, $\sigma(e) = e$.

Definition 4. The factor set associated with an extension E and a based section σ is the function $\varphi\colon G \times G \to A$ given by the rule:

$$\varphi(g, h) = \sigma(g)\sigma(h)(\sigma(gh))^{-1}.$$

Problem 6. Show that a factor set is well-defined, i.e. $\varphi(G \times G) \subset A$.

Problem 7. Let φ be the factor set associated with an extension E and a based section σ. Let E' be an equivalent extension. Show that there exists a based section $\sigma'\colon G \to E'$ such that the associated factor set coincides with φ.

Problem 8. A function $\varphi\colon G \times G \to A$ is a factor set if and only if $\varphi \in C^2(G, A)$ and $\delta^3(\varphi) = 0$.

Hint: show that bivalent operations on the set $A \times G$ that extend the multiplication on G by the multiplication on A corresponds to functions from $G \times G$ to A. Then show that under this correspondence the associativity turns into $\delta^3(\varphi) = 0$.

Problem 9. Let φ be the factor set associated with an extension E and a based section σ and let ψ be the factor set associated with the same extension E but with a different based section σ'. Show that $\varphi - \psi \in \mathrm{Im}(\delta^2)$. This means that there exists a function $\alpha\colon G \to A$, such that for all $g, h \in G$:

$$\varphi(g, h) - \psi(g, h) = g\alpha(h) - \alpha(gh) + \alpha(g).$$

The three last problems show that we have a well-defined surjective map from the set of equivalence classes of extensions to $H^2(G, A)$.

Problem 10. Show that this map is injective.

Hopf's Formula

Suppose that the group G is given by a corepresentation $G = \frac{F}{R}$, where F is a free group and R is a normal subgroup of F. One can ask the question how to compute $H_*(G, \mathbb{Z})$ in terms of the given corepresentation? Here we give the answer for this question when $* = 1, 2$.

Problem 11. Using the Hurewicz theorem $H_1(G, \mathbb{Z}) = \frac{G}{[G,G]}$, show that $H_1(G, \mathbb{Z}) = \frac{F}{R[F,F]}$.

Problem 12. Suppose that $F_n \to F_{n-1} \to F_{n-2} \to \ldots F_0 \to \mathbb{Z}$ is an exact sequence such that all F_i are projective $\mathbb{Z}[G]$-modules. Show that

1. $H_i(G, \mathbb{Z}) \cong H_i(F_G)$ for all $i < n$;
2. There exists the exact sequence of abelian groups:

$$0 \to H_{n+1}(G, \mathbb{Z}) \to H_n(F)_G \to H_n(F_G) \to H_n(G, \mathbb{Z}) \to 0.$$

The sequence $F_n \to F_{n-1} \to F_{n-2} \to \ldots F_0 \to \mathbb{Z}$ from the previous problem is called **a partial resolution** of the trivial G-module $\mathbb{Z}$.

Let Y be the wedge of circles such that $\pi_1(Y) \cong F$. Let $\tilde{Y}$ be the covering of Y corresponding to the normal subgroup R. Then G acts on $\tilde{Y}$ and the complex of singular chains

$$C_1(\tilde{Y}) \to C_0(\tilde{Y}) \to 0$$

is a partial resolution of $\mathbb{Z}$. This means that $H_2(G, \mathbb{Z}) \cong \ker(H_1(\tilde{Y})_G \to H_1(Y))$.

Problem 13.

1. Show that the F-action on R by conjugations induces the G-action on R_{ab}.
2. Show that $H_1(\tilde{Y})$ is isomorphic to R_{ab} as a G-module.
3. Show that $(R_{ab})_G \cong R/[F, R]$.
4. (Hopf's theorem) Show that $H_2(G, \mathbb{Z}) \cong \frac{R \cap [R,F]}{[F,F]}$.