Bounded Cohomology Problem Session

Please skip to the last problem(which isn't there yet) first.

1.Summary of what we need for the problem session The cohomology of a group G with coefficients in a G-module M, $H^i(G,M)$, is the cohomology of the complex $C^*(G,M) = Hom_G(Q_*, M)$: here the G-module $Q_n := \mathbb{Z}[G]\otimes\mathbb{Z}[G]^{\otimes n}$ has the trivial action on $\mathbb{Z}[G]^n$ and the usual action on $\mathbb{Z}[G]$. $d(1\otimes g_1\otimes ...\otimes g_n) = g_1 \otimes (g_2 \otimes ...g_n) - g_1g_2 \otimes ....g_n + ...(-1)^{n+1}g_1 \otimes ...g_{n-1}$. Call the coboundary δ. [1].

Note that when M has a trivial G action, then $C^n(G,M)$ as an abelian group is the group of functions from $G^{\times n} \to M$. In what follows take $M = \mathbb{R}$ or $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$.

One defines bounded cohomology $H^*_b(G,M)$ by taking bounded functions from $G^{\times n} \to M$.

1.1.**bounded cohomology of 'nice/amenable groups'=0.** We will call a group G amenable if there is a bounded linear functional $m : C_1(G,\mathbb{R}) \to \mathbb{R}$ that is invariant under G. Here, $C_1(G,\mathbb{R})$ is given the sup norm. To prevent the zero function we have the condition $m(1_G) = 1$.

Exercise 1.1. *This will show that $H^i(G,M) = 0$, $i > 0$, when G is amenable. Turn the following into a proof: Suppose we are given $[f] \in H^n(G,M)$. Using m, kill a coordinate of the function $f' : G^{\times n} \to \mathbb{R}$. This gives a candidate for $h \in C^{n-1}(G)$ such that $\delta h = f$. Choose the right coordinate to kill by using the condition $\delta f = 0$.*

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(Aside) In lecture we then used this, and the long exact sequence in bounded cohomology, coming from $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, to show that $H^2_b(G,M) = 0$ when G is perfect. We used this to show that any action of G on S^1 must have a fixed point.

1.2.**Euler numbers.** Given a map $\alpha \in Homeo_+(S^1)$, there is a rotation that approximates α. It is defined by $lim\tilde{\alpha}^n/n$ where $\tilde{\alpha}$ is a lift to a homeomorphism of $\mathbb{R}$ with $\tilde{\alpha}(0) = [0,1)$. This number is called the rotation number.

Now note we have a map $SO(2) \to Homeo_+(S^1)$. This map is a homotopy equivalence because there is a map going the other way, given by the rotation number. Rotation numbers naturally live/come up in the context of bounded cohomology.

1.2.1.*detour.* In order to make use of this homotopy equivalence, we need to introduce bounded cohomology of spaces.

Definition 1.2. *Let X be a topological space. Give the singular chain group $C_i(X)$ a norm by making the basis elements, namely the singular simplices, have norm 1. Let $C^i_b(X)$ be the bounded linear functionals on $C_i(X)$. The cohomology of this chain complex is called $H^i_b(X)$.*

Exercise 1.3. *Mentally prove the homotopy invariance of bounded cohomology of spaces.*

One can also define $H^i_b(X,M)$ with coefficients in a $\pi_1(X)$ module M by taking the cohomology of the complex $Hom_{\pi_1(X),bdd}(C_i(\tilde{X}),M)$. Just like in the unbounded case $H^0_b(X,M) = M^{\pi_1(X)}$

In regular cohomology $H_i(G,-) = H_i(BG,-)$ because they're both the derived functors of invariants.

Exercise 1.4. *Give a version of this argument for bounded cohomology.*

1.2.2.*bdd euler classes.* Consider a topological circle bundle $S^1 \to E \to M$. If the classifying map $S^1 \to BHomeo_+(S_1)$ factors through $BHomeo_+(S_1)^{discretetopology}$, or equivalently factors through $B\pi_1(M)$ say that the bundle is flat. There is a canonical bounded cohomology class $d \in H^2_b(BHomeo_+(S_1)) = H^2_b(BS_1) = H^2_b(Homeo(S_1)) = H^2_b(S_1)$ coming from the extension $\mathbb{Z} \to \mathbb{R} \to S^1$ or equivalently from the extension $\mathbb{Z} \to intertwiningHomeo_+(\mathbb{R}) \to Homeo_+(S_1)$. Evidently if the bundle is flat, then pulling back d along the factored classifying map gives a bounded euler class $e_b \in H^2_b(\pi_1(M))$ or can further pull back to get $e_b \in H^2_b(M)$.

Exercise 1.5. *Its true(but I haven't given you enough tools to show) that when $M = S^1$ that this bounded euler class is the euler number, after identifying $H^2_b(S_1) = \mathbb{R}/\mathbb{Z}$. Give a plausibility argument that when the bundle comes from a vector bundle of rank 2, that this rotation number is just the rotation given by the monodromy.*

[1] I stole this from Nikolai