GROMOV'S POLYNOMIAL GROWTH THEOREM

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1. Introduction and basic facts

Our goal today is to understand when and how the geometry of a group informs its algebra. Given an arbitrary finitely generated group, its algebra may be difficult to access. On the other hand, if we have a group which is also a smooth manifold, we may study it using powerful tools like calculus.

Recall that if G is generated by a finite set S, then G is a metric space under the metric

$$d_S(g,h) = ||h^{-1}g||_s$$

where $|| - ||_S$ is the *S-length*, i.e. the minimum number of elements $S \cup S^{-1}$ required to write the element $h^{-1}g \in G$. Equivalently, it's the minimum number of steps to get from g to h in the Cayley graph of G.

This turns G from a set into a metric space. We can then ask about the "shape" of G. For example, we can talk about the number of ends of a group as seen earlier in the semester.

Let $B_S(id,n)$ be a closed ball of radius n around $id \in G$. How does the number of elements in this ball (which is finite since G is finitely generated) grow as n grows?

Definition 1.1. The *word growth function*

$$N_{G,S} : \mathbb{Z}_+ \to \mathbb{Z}_+$$

is defined by sending $n \mapsto |B_S(id,n)|$.

The question above can then be reformulated by asking about the asymptotic behavior of $N_{G,S}$.

Definition 1.2. Given $f,g : \mathbb{Z}_+ \to \mathbb{Z}_+$, we say that $f \preccurlyeq g$ if there exists $C > 1$ such that $f(n) \le C \cdot g(C \cdot n)$ for all n. We say that $f \asymp g$ if $f \preccurlyeq g$ and $g \preccurlyeq f$.

Example 1.3.
1. We have $n^\alpha \asymp n^\beta$ if and only if $\alpha = \beta$. Growth of the form n^r for r fixed is called *polynomial growth*.
2. We have $2^n \asymp b^n$ for any $b > 1$. Growth of the form b^n for fixed b is called *exponential growth*.

This notion of equivalence is not sensitive to quasi-isometric changes of the group G. This tells us the first two of the following properties of word growth:

Proposition 1.4. *The following are true:*
1. *If S, S' are two finite generating sets of G, then $N_{G,S} \asymp N_{G,S'}$.*
2. *If $H \le G$ has finite index, then $N_H \asymp N_G$.*
3. *If $H \le G$ or G surjects on to H, then $N_H \preccurlyeq N_G$.*
4. *If M is a closed Riemannian manifold and $G = \pi_1(M)$, then*

$$N_G \asymp (\textit{volume of a ball of radius } n \textit{ in } \tilde{M}).$$

Example 1.5. Let M be a torus, so the universal cover $\tilde{M} \cong \mathbb{R}^2$ is $\mathbb{R}^2$. The covering map is the quotient by the integer lattice. A ball of radius 2 in the word metric in the group $\pi_1(M) = \mathbb{Z}^2$, with the standard generating set, consists of all the lattice points inside a diamond with vertices at $(0, \pm 2)$ and $(\pm 2, 0)$. Comparing to the Euclidean ball of radius 2, we see that the number of points enclosed is the same.

Remark 1.6. *Note that the last equivalence implies that the growth rate N_G depends only on the universal cover. In particular, this will not distinguish between a genus two surface and a genus three surface.*

Corollary 1.7. *If M is a closed Riemannian manifold of negative sectional curvature, then $\pi_1(M)$ has exponential growth.*

Roughly speaking, this follows from the observation that balls in hyperbolic space grow exponentially quickly in the radius.

2. Examples

Example 2.1. Let $G = \mathbb{Z}^r$. This has growth $N_G(n) \asymp n^r$.

Example 2.2. Let G be a free group on 2 generators (or finitely many generators). Then $N_G(n) \asymp 3^n$. This follows from a similar argument to what we saw in one of the first few problem sessions.

Example 2.3. Let G be the Heisenberg group, the group of matrices

$$G = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \beta, \gamma \in \mathbb{Z} \right\}.$$

Parametrize elements of G by triples (α, β, γ) and let $x = (1, 0, 0)$, $y = (0, 1, 0)$ and $z = (0, 0, 1)$. Then G has presentation

$$G = \langle x, y, z | [x, y] = z, zx = xz, zy = yz \rangle,$$

where $[x, y] = xyx^{-1}y^{-1}$. Take $S := \{x, y, z\}$ to be the generating set. You can check using matrix multiplication that $[x^a, y^b] = z^{ab}$. From this it follows that words of length on the order of $6n$ can produce elements of the form z^k for $-n^2 \leq k \leq n^2$. We conclude (see problem session for details) that $B_G(id, 8n)$ contains $w = x^a y^b z^c$ for $-n \leq a, b \leq n$ and $-n^2 \leq c \leq n^2$. Since these are distinct elements of G, we see that $n^4 \preccurlyeq N_G(n)$. The opposite inequality is also true, so $N_G(n) \asymp n^4$.

Example 2.4. Let $G = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ where $\mathbb{Z}$ acts by the matrix $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Equivalently, G is the set of 3×3 matrices with M^γ in the lower right, 0's above that, and $1, \alpha, \beta$ along the right-hand column, where $\alpha\beta, \gamma \in \mathbb{Z}$. We will show in problem session, using the fact that M has an eigenvalue off the unit circle, that G has exponential growth.

Remark 2.5. *The previous two examples are fundamental objects in 3-dimensional geometry. The Heisenberg group acts cocompactly on Thurston's Nil geometry, while $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ acts cocompactly on Sol geometry.*

3. Generalities

Recall that if $H, K \leq G$, then their commutator is the normal subgroup

$$[H, K] := \langle \{[h, k] : h \in H, k \in K\} \rangle \trianglelefteq G.$$

A group G has a *lower central series* defined inductively by

$$G_0 := G, \quad G_{k+1} := [G, G_k].$$

Sometimes this is interesting, sometimes it's not. For example, we say that G is *perfect* $G_i = G$. For example, this happens when G is simple. On the other hand, we say that G is *nilpotent* if $G_L = \{id\}$ for some L. In this case, the *nilpotence class* of H is the smallest L such that $G_L = \{id\}$.

Example 3.1. Abelian groups are nilpotent of class 1.

Example 3.2. The Heisenberg group is nilpotent of class 2. We have $G_1 = \langle z \rangle$ and $G_2 = \{e\}$.

Theorem 3.3 (Wolf, Bass, Guivarc'h)**.** *Any nilpotent group G has polynomial growth $N_G(n) \preccurlyeq n^r$ for some r. In fact, $N_G(n) \asymp n^{f(G)}$ where*

$$f(G) = \sum_{k=0}^{\infty}(k+1)\cdot \dim(G_k/G_{k+1}).$$

We say that G is solvable if the *derived series* defined by

$$G_{(0)} := G, \quad G_{(k+1)} := [G_{(k)}, G_{(k)}].$$

terminates. In particular, nilpotent groups are solvable. The converse does not hold - for example, the group $\mathbb{Z}^2 \rtimes_M \mathbb{Z}$ in the previous section is solvable but not nilpotent.

Theorem 3.4 (Milnor-Wolf)**.** *A finitely generated solvable group has polynomial growth if and only if it's nilpotent. Otherwise, it has exponential growth.*

Gromov's polynomial growth theorem eliminates the need for the 'solvable' hypothesis, providing a general converse to the theorem of Wolf.

Theorem 3.5 (Gromov)**.** *A finitely generated group G has polynomial growth if and only if G is* virtually nilpotent*, i.e. G has a nilpotent subgroup of finite index.*

In general, knowledge of the growth of G can only provide information about G up to finite-index, so the conclusion that G is only nilpotent up to finite index cannot be strengthened. Gromov's theorem provides really strong group-theoretic information about G; as such, the theorem has a really difficult proof!

Remark 3.6. *One corollary of Gromov's theorem, using the formula of Bass and Guivarc'h, is that if $N_G(n) \preccurlyeq n^r$ for some positive real number r, then $N_G(n) \asymp n^k$ for some positive integer k.*

Corollary 3.7. *Any group quasi-isometric to a nilpotent group is virtually nilpotent.*

Corollary 3.8 (Paulin)**.** *Any group quasi-isometric to an abelian group is virtually abelian.*

This requires the study of something called "asymptotic cones."

Corollary 3.9. *If a compact Riemannian manifold M admits an* expanding map*, i.e. a map $M \to M$ which scales the length of all tangent vectors by at least some uniform $\lambda > 1$, then M is finitely covered by a nilmanifold, i.e. a simply connected nilpotent Lie group modulo a subgroup.*

Corollary 3.10. *A random walk on a group is recurrent if and only if that group is $\mathbb{Z}^k$ for $k \in \{0, 1, 2\}$.*

4. Quick idea of the proof

Assume that G has polynomial growth. Then there are two steps:

(1) (Big step) Show that G has a finite dimensional representation $\rho : G \to GL_n\mathbb{R}$ with infinite image.
(2) Induct on the degree of growth using Tits's alternative: any finitely generated linear group is either virtually solvable or contains a nonabelian free subgroup.

Remark 4.1. *Kleiner and Tao–Shalom have given alternative proofs which use less difficult machinery.*

How does Gromov prove Step 1? He invents Gromov-Hausdoff convergence of metric spaces. Note that G acts on its Cayley graph (Γ, d_S). Let $X_n := (\Gamma, \frac{1}{n}d_S)$. Then X_n converges to some space Y with a G-action. One then shows (through many technical arguments and Montgomery–Zippin's solution to Hilbert's Fifth Problem) that (Y) is a Lie group, after passage to an sufficiently nice subsequence of X_n.