PROBLEMS ON WORD GROWTH OF GROUPS

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For a group G generated by a finite set S, let $\|g\|_S$ be the minimum length of a word in $S \cup S^{-1}$ that represents $g \in G$. The group G is a metric space with $d_S(g,h) = h^{-1}g$. Recall that the (word) growth of G is the function

$$N_{G,S}(n) := \#\{g \in G : \|g\|_S \leq n\}.$$

(1) Suppose G and H are groups generated by finite sets S and T, respectively. Show that if (G, d_S) and (H, d_T) are quasi-isometric metric spaces, then $N_{G,S} \asymp N_{H,T}$.

The Heisenberg group is

$$H_3 = \langle x, y, z | [x,y] = z, zx = xz, zy = yz \rangle$$

where $[x,y] := xyx^{-1}y^{-1}$. Consider H_3 with generating set $S = \{x, y, z\}$. A warmup exercise is to prove that any element of H_3 can be uniquely written as a word $x^k y^\ell z^m$.

(2) Show that $\|z^m\| \leq 6\sqrt{m}$. Conclude $N_{H_3,S}(n) \succcurlyeq n^4$.
(3) Show that if $\left\|x^k y^\ell z^m\right\|_S \leq r$ then $|k| + |\ell| \leq r$ and $|m| \leq r^2$. Conclude $N_{H_3,S}(n) \preccurlyeq n^4$.

Recall our basic example of a non-nilpotent solvable group $G = \mathbb{Z}^2 \rtimes_M \mathbb{Z}$ where $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Elements of G are tuples (α, β, γ), where $\alpha, \beta, \gamma \in \mathbb{Z}$. Let $a = (1,0,0)$, $b = (0,1,0)$ and $t = (0,0,1)$. (So G has the relations $tat^{-1} = a^2b$ and $tbt^{-1} = ab$.) Then $S = \{a, b, t\}$ generates G.

(4) Suppose $A \in GL_d(\mathbb{Z})$ is a matrix with an eigenvalue of complex norm at least 2. Show there is a vector $v \in \mathbb{Z}^d$ such that for any $k \geq 0$, the vectors

$$\epsilon_0 v + \epsilon_1 A(v) + \epsilon_2 A^2(v) + \cdots + \epsilon_k A^k(v)$$

are all distinct as $(\epsilon_i)_{i=0}^k$ takes all values in $\{0,1\}^{k+1}$.

(5) Show that the solvable group G above has $N_{G,S}(n) \succcurlyeq 2^n$, and conclude that $N_{G,S}(n) \asymp 2^n$.

The solvable group G is *polycyclic*; it has a subnormal sequence of subgroups $1 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G$ such that $G_k/G_{k-1} \cong \mathbb{Z}$ for all k. Solvable groups are generally more wild. A simple example are the solvable *Baumslag-Solitar groups*, defined by presentation

$$BS(1,n) = \left\langle a, b \mid aba^{-1} = b^n \right\rangle.$$

(6) Sketch the Cayley graph of $BS(1,2)$.
(7) Show that $BS(1,n)$ has exponential growth.

Being very clever about generalizing the above arguments proves the theorem of Milnor and Wolf, that any solvable group has either polynomial or exponential growth, and has polynomial growth exactly when it is nilpotent. Now let's talk about the easier part of the proof of the hard direction of Gromov's theorem.

Suppose G has polynomial growth, i.e. $N_G(n) \preccurlyeq n^r$ for some $r > 0$. Suppose Gromov has done the hard work of providing you with a linear representation $\phi : G \to GL_d(\mathbb{R})$ with infinite image, for some d.

(8) Show that $\phi(G)$ has a finite-index subgroup that is solvable. (You may want to use the following relevant theorem of Tits: any finitely generated linear group either is virtually solvable or contains a nonabelian free subgroup.)
(9) Show that there is a surjective homomorphism $f : G \to \mathbb{Z}$.
(10) Let K be the kernel of the map f of the previous problem. Show that if $N_G(n) \preccurlyeq n^r$ then $N_K(n) \preccurlyeq n^{r-1}$.
(11) Finish the proof by induction.