$$\begin{aligned} \mathrm{cd}\, G &= \mathrm{proj\,dim}_{\mathbb{Z}G}\, \mathbb{Z} \\ &= \inf\{n : \mathbb{Z} \text{ admits a proj. res. of length } n\} \\ &= \inf\{n : H^i(G,-)=0 \text{ for } i>n\} \\ &= \sup\{n : H^n(G,M) \neq 0 \text{ for some } M\} \end{aligned}$$

Bonus fact: if $\mathrm{cd}\, G = n$, then any partial proj. resolution of length $n-1$ can be completed to a proj. resolution of length n

If Y is a $K(G,1)$ complex, and X is its universal cover, then $H^*(G,M) = H^*(Y,M)$ bc $C_*(X)$ is a free res. of $\mathbb{Z}$ over $\mathbb{Z}G$

geom dim G = min dim of a $K(G,1)$ complex

$\mathrm{cd}\, G \leq$ geom dim G

<u>Examples</u> $\mathrm{cd}\, G = 0$ iff G is trivial
$\mathrm{cd}\, G = 1$ iff G is free and nontrivial

Can determine $\mathrm{cd}\, G=n$ using two existences:

- M s.t. $H^n(G,M) \neq 0$
- n-dim $K(G,1)$

A surface (closed, conn'd) Y is a $K(G,1)$ for $G=\pi_1(Y)$
so $\mathrm{cd}\, G \leq 2$. $H^2(G,\mathbb{Z}_2) = H^2(Y,\mathbb{Z}_2) \neq 0$
so $\mathrm{cd}\, G = 2$.

$G=\mathbb{Z}^n$: T^n is a $K(G,1)$, and $H^n(T^n,\mathbb{Z}) \neq 0$, so $\mathrm{cd}\, G = n$

Hochschild-Serre Spectral sequence:

$\forall\ 1 \to K \to G \to Q \to 1 \quad \forall\ G$-module M,

$\exists$ SS s.t.

$$E_2^{pq} = H^p(Q, H^q(K, M)) \Rightarrow H^{p+q}(G, M)$$

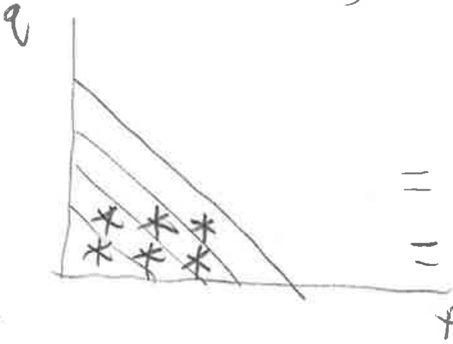
Let G = Heisenberg group

$Z = \left\{ \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \subseteq G \qquad 1 \to Z \to G \to G/Z \to 1$

So we have $E_2^{pq} = H^p(G/Z, H^q(Z, M)) \Rightarrow H^{p+q}(G, M)$

Note $Z \approx \mathbb{Z}$

$G/Z \approx \mathbb{Z} \oplus \mathbb{Z}$



$H^3(G, \mathbb{Z})$

$= H^2(G/Z, H^1(Z, \mathbb{Z}))$

$= H^2(G/Z, \mathbb{Z}) = \mathbb{Z}$

$M = \mathbb{Z}$

Central is important! $\quad \text{cd}\, G = 3$

- Proof generalizes to f.g. torsionfree nilpotent groups

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a) if $G' \subseteq G$, then $\text{cd}\, G' \leq \text{cd}\, G$

w/ eq. if $\text{cd}\, G < \infty$ and finite index

b) $1 \to G' \to G \to G'' \to 1$

$\text{cd}\, G \leq \text{cd}\, G' + \text{cd}\, G''$

Shapiro's Lemma

If G has torsion, $\text{cd}\, G = \infty$

finite cyclic $G' \subseteq G$ has $H^{2k}(G', \mathbb{Z}) \neq 0$

$\text{cd}\, G < \infty$ necessary, but only because of torsion

Serre's Theorem: If G torsion free, $G' \subseteq G$ finite index, then $\operatorname{cd} G = \operatorname{cd} G'$

Just need $\operatorname{cd} G' = n \Rightarrow \operatorname{cd} G < \infty$

By a later result: $\exists$ finite dim $K(G',1)$ complex

Let X' be its universal cover

Let $X = \operatorname{Hom}_{G'}(G, X')$ $\qquad G' \curvearrowright G$ left translation

$f \in X$, $g_0, g \in G$

$G \curvearrowright X$ by $(g_0 \circ f) : g \longmapsto f(g g_0)$

☆ So X has a G action

Choose coset reps $g_1, \ldots, g_n$ of $G' \backslash G$

Get $\varphi : X \longrightarrow \prod_{i=1}^{n} X'$ bijection

$f \longmapsto (f(g_1), f(g_2), \ldots, f(g_n))$

☆ so X is a finite CW complex

independent of choice

☆ also contractible

$g_0 f \longmapsto (f(g_1 g_0), \ldots, f(g_n g_0))$

Show free: $X \to X'$ eval at $1 \in G$

G' equivariant

G' acts freely on X

finite stabilizers $\Rightarrow$ trivial stabilizers

Eilenberg-Ganea: G arbitrary, $n = \max\{cd\,G, 3\}$, $\exists$ n-dim'l $K(G,1)$ complex

combined w/ Stallings-Swan ($n \geq 3$) and the easy $n=0$ case:

$cd\,G = geomdim\,G$, unless possibly

$cd\,G = 2$ and $geomdim\,G = 3$

No known example

Conjecture (Eilenberg-Ganea): there is no example

if $cd\,G = 2$, $\exists$ 2-dim $K(G,1)$ complex

Fun Fact: Bestvina/Brady found a group G s.t. either G is a counterexample, or the Whitehead conjecture is false

(Every connected subcomplex of a 2-dim aspherical CW complex is aspherical)

Virtual cohomological dimension

If G virtually torsion free, then all torsion free G' of finite index have same cd

$G' \supseteq G' \cap G'' \subseteq G''$ and Serre's theorem

So we can define $vcd\,G = cd\,G'$ for any torsion free G' of finite index in G

$SL_2(\mathbb{Z})[m] \leq SL_2(\mathbb{Z})$ for $m \geq 3$ is free w/ finite index, so

$vcd(SL_2(\mathbb{Z})) = 1$

$vcd(SL_n(\mathbb{Z})) = \frac{n(n-1)}{2}$

Other results: Suppose Y is a d-dim. $K(G,1)$ manifold.
$cd(G) \leq d$, equality iff Y is closed

Other definitions: FP_∞: $\exists$ proj res. of finite type

FP: $cd\, G < \infty$ and G is of type FP_∞

FL: finite free resolution

If G is finitely presented and type FL (FP)
then $\exists$ finite (finitely dominated) $K(G,1)$ complex

type FP, $n = cd\, G$, then
$$H^n(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \longrightarrow H^n(G, M)$$
is an iso $\forall M$

$[G : G']$ finite, G is type $FP \Leftrightarrow G'$ type FP

If G is of type FP, then $cd\, G = \max\{n : H^n(G, \mathbb{Z}G) \neq 0\}$

If G is of type FP, $n = cd\, G$, then
$$\varphi : H^n(G, \mathbb{Z}G) \otimes_{\mathbb{Z}G} M \xrightarrow{\cong} H^n(G, M)$$

Map is $u \in Hom_G(P, \mathbb{Z}G)$, $m \in M$, send $u \otimes m$ to
$x \mapsto u(x)m$

FL $\Rightarrow$ finite complex

FP $\Rightarrow$ finitely dominated complex

(is a homotopy retract of a finite complex)

$$\mathrm{WFL} \Rightarrow \mathrm{VFL} \Rightarrow \mathrm{VFP} \Rightarrow (\mathrm{vcd} < \infty)$$

WFL $\Updownarrow$ virtually torsionfree and <u>every</u> is FL

VFL $\Updownarrow$ <u>some</u> finite index is FL

VFP $\Updownarrow$ <u>some</u> finite index is FP $\Downarrow$ every is