$(G/H)_+ \wedge D^n$?

<u>Recall:</u>

- A $G$CW-complex is inductively built by attaching $G$-cells $(G/H \times D^n)$ along a $G$-map $G/H \times S^{n-1} \longrightarrow X^{\langle n-1 \rangle}$, which is determined by the map
$$S^{n-1} \longrightarrow (X^{\langle n-1 \rangle})^H$$

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based $\mathcal{U}$-

In $\mathcal{T}$, $f: X \to Y$ an $n$-equivalence if $\pi_q(f): \pi_q(X) \longrightarrow \pi_q(Y)$ is a bijection for $q<n$ and a surjection if $q=n$.

- What is an $n$-equivalence in $G\mathcal{U}$ or $G\mathcal{T}$?

<u>Homework</u>: If $H_1$ and $H_2$ are conjugates then
- $X^{H_1} \cong X^{H_2}$
- $X/H_1 \cong X/H_2$

A number $n$ can be thought of as a map
$$\underline{n}: \{*\} \longrightarrow \mathbb{Z}$$
$$* \longmapsto n.$$

▶ Replace $*$ by conjugacy classes of subgroups of $G$, $\mathcal{C}G$.

Let $\nu: \mathcal{C}G \longrightarrow \mathbb{Z}_{\geq -1}$. We say that a map $e: Y \to Z$ is a $\nu$-equivalence if
$$e^H: Y^H \longrightarrow Z^H$$
is a $\nu(H)$-equivalence for all $H$.

▶ We say that a ~~map~~ $G$-CW complex has dimension $\nu$ if the cells of <u>orbit type $G/H$</u> all have dimension <u>less than or equal to $\nu(H)$</u>.

**Theorem** (HELP): Let $A$ be a subcomplex of a $G$-CW-complex $X$ of dimension $\nu$ and let $e: Y \to Z$ be a $\nu$-equivalence. Suppose given maps $g: A \to Y$, $h: A \times I \to Z$ and $f: X \to Z$ such that $\boxed{eg = hi_1}$ and $\boxed{fi = hi_0}$ in the diagram

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll] & \\
X \arrow[rr] \arrow[ur, "f"] & & X \times I \arrow[ul, dashed, "\tilde{h}"] & & X \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"]
\end{tikzcd}$$

**Proof**: "Usual" (Homework).

**Theorem** (Whitehead): Let $e: Y \to Z$ be a $\nu$-equivalence and $X$ be a $G$-CW complex. Then $e_*: hG\mathcal{U}(X, Y) \to hG\mathcal{U}(X, Z)$ is a bijection if $X$ has dimension less than $\nu$ and surjection if $X$ has dimension $\nu$.

**Corollary**: If $e: Y \to Z$ is a $\nu$-equivalence between $G$-CW complexes of dimension less than $\nu$, then $e$ is a $G$-homotopy equivalence.

**Proof**: By Whitehead's Theorem,

$$e_*: hG\mathcal{U}(Z, Y) \to hG\mathcal{U}(Z, Z)$$
$$f \longmapsto 1_Z$$

Argue ~~Claim~~: $f$ is homotopy inverse of $e$.

Theorem (Cellular Approximation)

Let $X$ and $Y$ be a $G$CW-complex. Let $X'$ be a subcomplex, ~~then~~ and $f : X \to Y$ be $G$-map whose restriction to $X'$ is cellular. Then $f \simeq g$ rel $X'$ such that $g$ is cellular.

Thm: For any $G$-space $X$, there is a $G$-CW complex $\Gamma X$ and a weak equivalence $\gamma : \Gamma X \to X$.

- $\Gamma : h\mathcal{G}\mathcal{U} \to h\mathcal{G}\mathcal{U}$ is a functor such that $\Gamma$ is natural.

- $$\underbrace{h\mathcal{G}\mathcal{U}(X, X') = h\mathcal{G}\mathcal{U}(\Gamma X, \Gamma X') = h\mathcal{G}\mathcal{C}(\Gamma X, \Gamma X')}_{\text{new name}}$$

$$[X, X']_G.$$

Ordinary homology and cohomology theories

- $\mathcal{G}$ - cat of orbit $G$-spaces.
  - Obj: $G/H$
  - Mor: $G/H \xrightarrow{f} G/K$, where $f$ is a $G$-map
    - $f(eH) = gK$, well defined if $g^{-1}Hg \subset K$.

distinction is important

- $h\mathcal{G}$ - the homotopy category of $\mathcal{G}$.

⊙ Define a coefficient system to be a contravariant functor
$$h\mathcal{G} \to \mathcal{A}b.$$

- Example 1: $\underline{\pi_n}(X)$
$$\underline{\pi_n}(X)(G/H) = \pi_n(X^H).$$

Example 2: $X^*: \mathcal{G} \to \mathcal{T}$, be the fixed point functor, where $X \in \mathcal{T}$.
$$X^*(G/H) = X^H$$
$$f: G/H \to G/K \quad \text{induces } f^*: X^K \to X^H$$
Let $F: h\mathcal{T} \to \mathcal{A}b$ covariant functor, then
$$h\mathcal{G} \xrightarrow{hX^*} h\mathcal{T} \longrightarrow \mathcal{A}b$$
is a coefficient system.

Example 3: Let $X$ be a GCW-complex. Define $[\cancel{K}\, g^{-1}Hg \subset K]$
$$\underline{C_n}(X) = \underline{H_n}(X^n, X^{n-1}; \mathbb{Z})$$
collection of all $G/K$ cells of dim $n$.
$$\cancel{\&}\ \cancel{\text{is}} \quad \underline{C_n}(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H; \mathbb{Z})$$
$d$ be the connecting homomorphism of triple $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$
$$d^2 = 0 \quad ((X^{n-1})^H, (X^{n-2})^H) \longrightarrow ((X^n)^H, (X^{n-2})^H) \longrightarrow ((X^n)^H, (X^{n-1})^H)$$

$d$

$H_n((X^n)^H, (X^{n-1})^H)$

$\longrightarrow H_{n-1}((X^{n-1})^H, (X^{n-2})^H)$

## Bredon Cohomology

⊙ Let $\mathrm{Hom}_{\mathcal{G}}(M, M')$ for the abelian group of maps of coefficient systems $M \to M'$ and define

$$C^n_G(X;M) = \mathrm{Hom}_{\mathcal{G}}(\underline{C_n}(X);M) \text{ with } \delta = \mathrm{Hom}_{\mathcal{G}}(d, id)$$

Then $C^*_G(X;M)$ is a cochain complex of Abelian groups. Its homology is the Bredon Cohomology of X.

⊙ To define Bredon homology of X, ~~define~~ consider covariant functor

$$N : h\mathcal{G} \to Ab.$$

If M is contravariant, define

$$M \otimes_{\mathcal{G}} N = \sum_{H<G} M(G/H) \otimes N(G/H) \Big/ \sim$$

$$[\,(mf^*, n) \sim (m, f_* n) : f : G/H \to G/K\,]$$

Define $C_n^G(X;N) = \underline{C_n}(X) \otimes_{\mathcal{G}} N$ with $\partial = d \otimes 1$.

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## Eilenberg Mclane space

Let M be a coeffiecient system. An Eilenberg-Mclane G-space $K(M,n)$ is a G-space of homotopy type of a GCW-complex such that

$$\underline{\pi_q}(K(M,n)) = \begin{cases} M & \text{if } q = n \\ 0 & \text{if } q \neq n. \end{cases}$$

$\underline{\text{Thm}}$ : $\tilde{H}^n_G(X;M) = [X, K(M,n)]_G.$