MACKEY FUNCTORS

References:
Hill-Hopkins-Ravenel, "Equivariant stable homotopy and the Kervaire invariant one problem"
Hill-Hopkins-Ravenel, "On the nonexistence of elements of Kervaire invariant one problem"
May, "Alaska Notes"
Talbot 2016 notes

1. Definition and some classical examples

We saw two weeks ago that a coefficient system is a functor from the homotopy category of G-orbits to the category of abelian groups. Today we'll introduce further structure to this picture which will be important in future weeks. From now on let G be a finite group; one can make sense of these ideas for compact Lie groups, but we won't need this for the Kervaire problem so it can be left to the interested listener.

We'll start with an enhanced version of the homotopy category of G-orbits. Let $\mathcal{F}^G$ denote the category of finite G-sets.

Definition 1.1. The *Lindner category* $\mathcal{B}_G^+$ for a finite group has objects finite G-sets. For G-sets X and Y, morphisms $X \to Y$ are equivalence classes of diagrams in $\mathcal{F}^G$ of the form

$$X \leftarrow U \rightarrow Y$$

where two diagrams (called *spans*) are equivalent if there is an isomorphism between the middle object. The morphism set $\mathcal{B}_G^+(X, Y)$ is an abelian monoid under disjoint union of middle objects with zero morphism where $U = \emptyset$.

The composition is defined as follows. Let $Y \leftarrow V \rightarrow Z$ represent the morphism $Y \to Z$. Then the composite morphism $X \to Z$ is represented by $X \leftarrow W \rightarrow Z$ where $W = U \times_Y V$ is the pullback of the square in the diagram

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr] & & \\
 & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

Definition 1.2. The *Burnside category* $\mathcal{B}_G$ for a finite group G has the same objects as the Lindner category, but the morphism set is the Grothendieck group completion of the abelian monoid $\mathcal{B}_G^+(X, Y)$ with composition induced by the composition in $\mathcal{B}_G^+$.

Definition 1.3. A *Mackey functor* $\underline{M}$ for a finite group G is an additive functor $\mathcal{B}_G^+ \to Ab$, i.e. a functor enriched over abelian monoids which sends disjoint unions to direct sums.

Equivalently, it is an additive functor from $\mathcal{B}_G \to Ab$, i.e. a functor enriched over abelian groups which sends disjoint unions to direct sums.

Exercise: Show these two definitions are equivalent.

Remark 1.4. *Note that any finite G-set decomposes as a disjoint union of orbits of the form G/H; since we'll require a Mackey functor to be additive, this implies that the value of a Mackey functor is determined by its values on orbits of this form.*

Example 1.5. *The* Burnside ring $A(G)$ *of a group G is the Grothendieck group of the abelian monoid (under disjoint union) of isomorphism classes of finite G-sets with multiplication induced by Cartesian product.*

The Burnside Mackey functor $\underline{A(G)}$ *for a group G is defined by setting* $\underline{A(G)}(X) := \mathcal{B}_G(G/G, X)$.

Exercise: *Understand what this Mackey functor does on morphisms. We'll see below that a Mackey functor has transfer and restriction maps; for the Burnside Mackey functor, transfer is given by composition and restriction is given by pullback.*

This definition is concise but can obfuscate some of the structure of Mackey functors. The following equivalent definition shines light on some of this structure.

Definition 1.6. A *Mackey functor* $\underline{M}$ for a finite group G is a pair of functors $M_* : \mathcal{F}^G \to Ab$ and $M^* : (\mathcal{F}^G)^{op} \to Ab$ that agree on objects, send finite disjoint unions to direct sums, and such that for every pullback diagram in $\mathcal{F}^G$

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"] & S \arrow[d, "\gamma"] \\ T \arrow[r, "\delta"] & U \end{tikzcd}$$

we have $M^*(\gamma)M_*(\delta) = M_*(\alpha)M^*(\beta)$. For a finite G-set T define

$$\underline{M}(T) := M^*(T) = M_*(T)$$

and for subgroups $K \subset H \subset G$ with projection $p : G/K \to G/H$, denote the induced maps by $Tr_K^H = M_*(p)$ (transfer) and $Res_K^H = M^*(p)$.

Example 1.7. *Let $RO(G)$ be the ring of real orthogonal representations of G, i.e. the Grothendieck group of the abelian monoid under direct sum of isomorphism classes of finite dimensional orthogonal representations V of G with multiplication induced by tensor product.*

For $H \subset G$, there is a restriction map

$$Res_H^G : RO(G) \to RO(H)$$

obtained by restricting the action of G- to an H-action.

There is also a transfer map

$$Tr_H^G : RO(H) \to RO(G)$$

defined by setting

$$Tr_H^G W := \mathbb{R}[G] \otimes_{\mathbb{R}[H]} W.$$

Exercise: *Show that $RO(G)$ defines a Mackey functor $\underline{RO(G)}$ by setting $\underline{RO(G)}(G/H) = RO(H)$.*

2. More examples

Example 2.1. *The* constant Mackey functor $\underline{\mathbb{Z}}$ *is represented by $\mathbb{Z}$ with trivial action, i.e. it is defined by*

$$\underline{Z}(B) = Hom^G(B, \mathbb{Z}) = Hom(B/G, \mathbb{Z}).$$

Example 2.2. *Let $S \in \mathcal{F}^G$ be a finite G-set, and let $\mathbb{Z}\{S\}$ be the free abelian group generated by S. The* permutation Mackey functor *on S is the Mackey functor represented by $\mathbb{Z}\{S\}$, i.e. it is defined by*

$$\mathbb{Z}\{S\}(B) = Hom^G(B, \mathbb{Z}\{S\}).$$

Restriction maps are given by precomposition and transfer maps are given by summing over fibers, i.e. if $g : A \to B$ then $\mathbb{Z}\{S\}_*(g)(f)(b) = \sum_{x \in g^{-1}(b)} f(x)$.

Example 2.3. *This example shows that equivariant stable homotopy groups define a Mackey functor. If B is a finite G-set and X is a G-space, we can set*

$$(\underline{\pi}_n(X))^*(B) = [S^n \wedge B_+, X]^G$$

$$(\underline{\pi}_n(X))_*(B) = [S^n, B_+ \wedge X]^G.$$

These form a Mackey functor since finite G-sets are self-dual, i.e. there is an isomorphism

$$[G/H_+ \wedge E, F] \cong [E, G/H_+ \wedge F].$$

In this case, saying these form a Mackey functor is saying that equivariant stable homotopy groups come with some additional structure since evaluation on G/H gives

$$\underline{\pi}_n(X)(G/H) = [S^n \wedge G/H_+, X]^G = [S^n, X]^H = \pi_n^H(X).$$

Exercise: *Show that this defines a Mackey functor, i.e. understand restriction and transfers.*

Some properties of these Mackey functors will be developed in the exercises. We can visualize a Mackey functor using a Lewis diagram:

Example 2.4. *A* Lewis diagram *is a tool for understanding a Mackey functor. If the subgroups of G are linearly ordered, e.g. if G is cyclic of order p^k, then we can order the resulting quotients*

$$e = C_{p^k}/C_{p^k} \subset C_{p^k}/C_{p^{k-1}} \subset \cdots \subset C_{p^k}/C_p \subset C_{p^k}/e = C_{p^k}.$$

We then apply the Mackey functor and draw in the transfer and restriction maps. For example, we have [Lewis diagram for C_8].

Exercise: *For various C_2-actions on S^1, draw the Lewis diagram for $\underline{\pi_1}(S^1)$.*

3. Mackey functor homology

Recall from Prasit's second talk that we could define Eilenberg-Mac Lane G-spaces for any coefficient system M, denoted $K(M,n)$. These satisfied

$$\underline{\pi}_q(K(M,n)) = \begin{cases} M & q = n \\ 0 & q \neq n \end{cases}$$

and we saw that these represented reduced Bredon cohomology:

$$\tilde{H}^n_G(X;M) = [X, K(M,n)]_G.$$

We can define *Mackey functor homology* following the same line of thinking.

Proposition 3.1 (Greenlees-May)**.** *Given a Mackey functor $\underline{M}$, there is an equivariant Eilenberg-Maclane spectrum $H\underline{M}$ such that*

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Definition 3.2. Given a Mackey functor $\underline{M}$, we can define *equivariant homology with coefficients in $\underline{M}$* as

$$H^G_k(X;\underline{M}) = \pi^G_k(H\underline{M} \wedge X)$$

and *equivariant cohomology* as

$$H^k_G(X;\underline{M}) = [X, \Sigma^k H\underline{M}]^G.$$

This definition is nice but we'd like a chain complex as in the case of Bredon cohomology which allows us to understand equivariant (co)homology by getting our hands dirty.

Definition 3.3. Let X be a G-CW spectrum. Recall that

$$X^{(n)}/X^{(n-1)} \sim X_{n+} \wedge S^n$$

where X_n is a discrete G-set. Set

$$C^{cell}_n(X;\underline{M}) = \pi^G_n H\underline{M} \wedge X^{(n)}/X^{(n-1)} = \pi^G_0 H\underline{M} \wedge X_{n+}$$

$$C^n_{cell}(X;\underline{M}) = [X^{(n)}/X^{(n-1)}, \Sigma^n H\underline{M}]^G = [\Sigma^\infty X_{n+}, H\underline{M}]^G$$

The boundary maps

$$C^{cell}_n(X;\underline{M}) \to C^{cell}_{n-1}(X;\underline{M})$$

are defined using the map

$$X^{(n)}/X^{(n-1)} \to \Sigma X^{(n-1)}/X^{(n-2)}$$

and the coboundary maps are defined similarly.

The homology of these complexes are the equivariant homology and cohomology groups of X with coefficients in $\underline{M}$. By writing the G-set X_n as a coproduct of finite G-sets, one can express these complexes in terms of the values of $\underline{M}$ on these finite G-sets.

Example 3.4 (HHR Example 3.7)**.** *Let $X = S^{d-1}$ with action of C_{2^n} given by antipodal map, pointed by adding a disjoint basepoint. Hemisphere decomposition is equivariant for the action of C_{2^n}. One has*

$$X^{(j)}/X^{(j-1)} = (C_{2^n}/C_{2^{n-1}})_+ \wedge S^j$$

and the complex of cellular chains $C_*^{cell}(X;\underline{M})$ is therefore a complex of length d

$$\underline{M}(C_{2^n}) \to \cdots \xrightarrow{1+\gamma} \underline{M}(C_{2^n}) \xrightarrow{1-\gamma} \underline{M}(C_{2^n})$$

where γ is a generator for C_{2^n}.

Exercise: *Work out this example with $n = 3$ and $\underline{M} = \underline{\mathbb{Z}}$. More generally, show that $C_n^{cell}(X;\underline{\mathbb{Z}})$ is the permutation Mackey functor $\underline{\mathbb{Z}}\{X_n\}$ which associates to a finite G-set B the group of equivariant functions*

$$B \to \mathbb{Z}\{X_n\} = C_n^{cell}X.$$

Lemma 3.5 (HHR Example 3.9)**.** *If X is a G-space admitting the structure of a G-CW complex, then the equivariant cohomology groups $H_G^*(X;\underline{\mathbb{Z}})$ are isomorphic to the cohomology groups $H^*(X/G;\mathbb{Z})$ of the orbit space.*

Proof. The equivariant cell decomposition of X induces a cell decomposition of X/G so we obtain an isomorphism

$$C^*_{cell}(X;\underline{\mathbb{Z}}) \simeq C^*_{cell}(X)^G \simeq C^*_{cell}(X/G)$$

□

Exercise: Apply the lemma to show that if V is a representation of G of dimension d, then there is a (non-canonical) isomorphism

$$H_d^G(S^V;\underline{\mathbb{Z}}) = \mathbb{Z}.$$

4. The closed symmetric monoidal category of Mackey functors

Definition 4.1. The category of G-Mackey functors and natural transformations is denoted $\mathfrak{M}_G$.

The Day convolution is a powerful construction which allows one to combine two different symmetric monoidal structures to produce a symmetric monoidal structure on a certain functor category. For example, the smash product in many categories of structured spectra (e.g. symmetric spectra) can be formulated as a Day convolution.

Definition 4.2 (Day convolution)**.** Let $\mathcal{D} = (\mathcal{D}_0, \oplus, 0)$ be a small symmetric monoidal $\mathcal{V}$-category where $\mathcal{V} = (\mathcal{V}_0, \otimes, 1)$ is a cocomplete closed symmetric monoidal category, and let $X, Y \in [\mathcal{D}, \mathcal{V}]$ be $\mathcal{V}$-functors. Then we define $X \square Y$ to be the left Kan extension of $\otimes \circ (X \times Y)$ along $\oplus$,

$$\begin{tikzcd} \mathcal{D} \times \mathcal{D} \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{V} \times \mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\ & \mathcal{D} \arrow[ur, dashed] & \end{tikzcd}$$

Example 4.3. *Let $\mathcal{D} = \mathcal{J}_G$ be the category of finite-dimensional orthogonal representations with unit the 0-dimensional representation and product $\oplus$. We can think of a G-spectrum X as a functor $X : \mathcal{J}_G \to \mathcal{T}_G$ where $\mathcal{T}_G$ is the category of topological G-spaces and not necessarily equivariant maps. The category $\mathcal{T}_G$ is a cocomplete symmetric monoidal category under $\wedge$. Then the diagram*

$$\begin{tikzcd} \mathcal{J}_G \times \mathcal{J}_G \arrow[r, "X \times Y"] \arrow[dr, "\oplus"'] & \mathcal{T}_G \times \mathcal{T}_G \arrow[r, "\wedge"] & \mathcal{T}_G \\ & \mathcal{J}_G \arrow[ur, dashed, "X \wedge Y"'] & \end{tikzcd}$$

defines a G-spectrum which we call $X \wedge Y$ (in view of the following theorem). If we take $G = e$, then this recovers classical orthogonal spectra, and writing out the definition of left Kan extension shows that the n-th space of the smash product $(X \wedge Y)_n$ is the coequalizer

$$\Big(\bigvee_{p+1+q=n} O(n)_+ \wedge_{O(p)\times 1 \times O(q)} X_p \wedge S^1 \wedge X_q\Big) \rightrightarrows \Big(\bigvee_{p+q=n} O(n)_+ \wedge_{O(p)\times O(q)} X_p \wedge X_q\Big) \to (X \wedge Y)_n.$$

Theorem 4.4 (Day convolution theorem)**.** *The binary operation above gives the functor category $[\mathcal{D}, \mathcal{V}]$ a closed symmetric monoidal structure in which the unit element is the $\mathcal{V}$-functor $I = h^0$ given by $I_D = \mathcal{D}(0, D)$. The internal Hom-functor is the right adjoint of the functor $(-) \otimes X$.*

Example 4.5. *For a finite group G, the* box product $\underline{M} \square \underline{N}$ *of two G-Mackey functors is the left Kan extension*

$$\begin{tikzcd} \mathcal{B}_G \times \mathcal{B}_G \arrow[r, "\underline{M} \times \underline{N}"] \arrow[dr, "\times"'] & Ab \times Ab \arrow[r, "\otimes"] & Ab \\ & \mathcal{B}_G \arrow[ur, "\underline{M} \square \underline{N}"'] & \end{tikzcd}$$

This gives the category of G-Mackey functors $\mathfrak{M}_G$ the structure of a closed symmetric monoidal category with unit the Burnside Mackey functor $\underline{A}$.