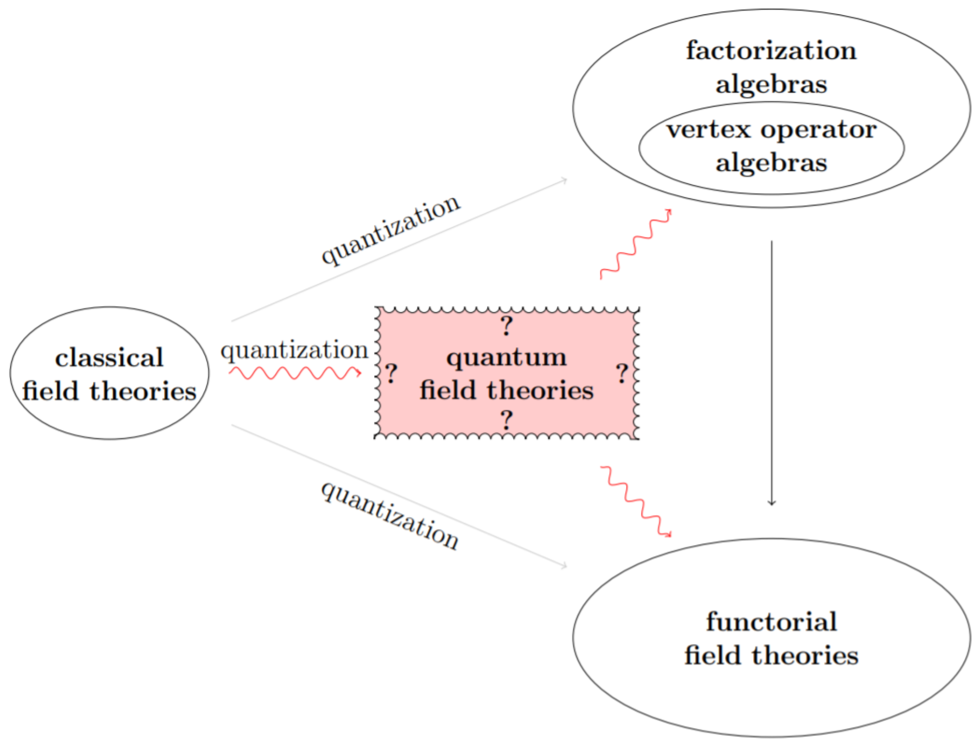
Notre Dame Graduate Student Topology Seminar, Spring 2018

Lecture 1: The definition of topological field theories and their physical motivation

1.1 Overview



Stephan has this nice schematic describing the lay-of-the-land that we will be discussing this semester.[1] Quantum field theories are mysterious objects in physics: physicists use them and have great predictive success, but even what QFT are is not well-defined. Whatever they are, they should somehow be related to classical field theories through a process know as 'quantization' (although this process is also just as mysterious; is has been called 'an art, not a science'). Over the years, mathematicians have proposed various ways of modeling or approximating quantum field theories, including functorial field theories (Segal, Atiyah and Kontsevich, 1980s), vertex operator algebras (Borcherds, 1986), and factorization algebras (Beilinson and Drinfeld, 2004). In this mini-course I'll be focusing on functorial field theories.

Before giving the definition of functorial field theories, I want to discuss the physical motivation behind the axioms chosen by Atiyah and Segal. Then I will give the mathematical definition of topological field theories (a specific type of functorial field theory, where we are only interested in the topological properties of spacetime, not additional geometric properties)

[1] *Functorial Field Theories and Factorization Algebras*, Stephan Stolz, course notes, Spring 2014, https://www3.nd.edu/ stolz/Math80440(S2014)/

and discuss some of the properties TFT have.

1.2 Physical Motivation

Let us start with a really simple physical system: a point particle moving in a Riemannian manifold X (with metric g).

1.2.1 Classical story

In classical mechanics, this point would be described by paths $\phi : [a, b] \to X$, where $\phi(t)$ gives the position of the particle at time $t \in [a, b]$. If there is a force field F acting on this particle, Newton's second law tells us this will affect the motion of the particle (with mass m) in the following way:

$$F = ma.$$

More precisely, we would write this

$$F(\phi(t)) = m\ddot{\phi}(t)(\in T_{\phi(t)}X).$$

Physicists talk about the collection of all such paths ϕ as the *space of fields*: $\mathcal{M} := C^\infty([a, b], X)$.

> The case we're talking about here has zero-dimensional space (because we're looking at the particle), and so the total dimension of space-time is 1 (hence the 1-dimensional interval). For a general spacetime M, one can look at the field theory whose space of fields is $\mathcal{M} = \{\phi : M \to X\}$; this field theory is called the *non-linear σ-model with target X.*

Physicists also often frame field theories in terms of an *action functional.* In the case of classical mechanics (the point particle moving under a force field F), this is called the Lagrangian formulation: you use the force field and Riemannian metric g to define the Lagrangian

$$L : TX \to \mathbb{R}$$
$$(x, v) \mapsto \frac{m}{2}||v||^2 - V(x)$$

They then write the action functional:

$$S : \mathcal{M} \to \mathbb{R}$$
$$S(\phi) := \int_a^b L(\phi(t), \dot{\phi}(t))dt$$

The critical points of the action functional are precisely the solutions to Newton's equation, $F(\phi) = m\ddot{\phi}$. (For details, see Stephan's notes, Thm 4.2 and following, pgs. 6-12.)

> In general, the *action functional* of a field theory will be a map from the space of fields,
>
> $$S : \mathcal{M} \to \mathbb{R}.$$
>
> For ϕ to be a critical point of S means that ϕ satisfies a differential equation (called the *Euler-Lagrange equation*) for the theory.

Classically, the *state* of a physical system at a fixed time is given by the position and velocity of the particle at that time. One question we could ask is how the state evolves over time: given the state at time $t = 0, (x_0 = \phi(0), v_0 = \dot{\phi}(0))$, what is the physical state at time $t, (x_t, v_t)$? The Riemannian metric on X gives a notion of 'geodesic flow': $(t\phi)(s) := \phi(t + s)$.

	physical state of the system	time evolution
Classical mechanics	point $(x, v) \in TX$	geodesic flow
Quantum mechanics	'wave function': unit vector $\psi \in L^2(X, \mathbb{C})$	$\psi_t = e^{-i\Delta t/\hbar}\psi_0$

1.2.2 Quantum story

Still looking at a point particle moving in a Riemannian manifold X, what happens in the quantum version? The picture looks very different. Here a physical state is described by a *wave function*: a square integrable function $\psi : X \to \mathbb{C}$ (i.e. $\psi \in L^2(X, \mathbb{C})$). Wave functions are notoriously difficult to understand. To get an intuition of what we're looking at, it's helpful to consider the n-form: $|\psi|^2 vol_g$. Because ψ is a unit vector:

$$\int_X |\psi(x)|^2 vol_g(x) = \int_X \overline{\psi(x)}\psi(x) vol_g(x) = \langle \psi, \psi \rangle = ||\psi||^2 = 1.$$

So we can think of $|\psi|^2 vol_g$ as a probability measure–the wave function doesn't tell us the position of the particle (because in quantum mechanics, that's not well-defined), but rather the probability that the particle is in a certain position.

The time evolution in quantum mechanics is given by the unitary operator $e^{-i\Delta t/\hbar}$. Here $\hbar$ is the Planck constant; Δ is the positive definite Laplace operator on X. In the 1940s, Feynman showed that one could write this time evolution as a sort of 'averaging over paths'– his famed *path integral*. It would look something like this:

$$(e^{-it\Delta/\hbar}\psi)(x) \text{``}=\text{''} \int_{\{\phi:[0,t]\to X | \phi(t)=x\}} \psi(\phi(0)) \frac{e^{-iS(\phi)/\hbar} D\phi}{Z}$$

<u>**Warning:**</u> This is not rigorously defined! It's not clear what the measure on the right hand side is, and also for higher-dimensional space-times, this integral doesn't make sense...

However, even though there are numerous difficulties with general path integrals, these have still proven quite fruitful as inspiration for what we're looking for in a quantum description of time evolution. In fact, the functorial field theories we'll be talking about are one of the methods inspired by the path integral. To more explicitly connect the path integral inspiration to TFT, I'm going to take a slight detour: instead of looking at the unitary operator $e^{-it\Delta/\hbar}$ of quantum mechanics, let's replace the *it* by t and set $\hbar = 1$. Then we'll be looking at the operator $e^{-t\Delta}$. This operator is called the *heat operator*; it is closely related to $e^{-}it\Delta/\hbar$ (by holomorphic extension), but much easier to understand.

Back in the 1940s, when Feynman was showing that the Schrodinger equation could be written as the 'path integral' that I mentioned above, Mark Kac (a mathematician at Cornell, where Feynman was at the time) came up with a similar formulation of the solutions to the heat equation. This formulation, as an 'averaging over paths', is known as the Feynman-Kac formula. It works in a similar way to the quantum mechanics that we were looking at: Let $f_0 : X \to \mathbb{R}$ describe the heat distribution at time $t = 0$. After time t has passed, the new heat distribution will be $f_t = e^{-t\Delta} f_0$. The Feynman-Kac formula states that this can be understood as an integral:

$$f_t(x) = (e^{-t\Delta} f_0)(x) = \int_{\{\phi:[0,t]\to X | \phi(t)=x\}} f(\phi(0)) \frac{e^{-S(\phi)}\mathcal{D}\phi}{Z}$$

Here

- $S(\phi)$= energy of the path ϕ
- $\mathcal{D}$=volume form
- Z=renormalization constant that makes $\frac{e^{-S(\phi)}\mathcal{D}\phi}{Z}$ into a probability measure: the Wiener measure

Interpretation: heat diffuses through material by the motion of molecules as they successively bump into each other. The temperature at time t at a point x is affected by the temperature at time $t = 0$ at all points whose motion reached x by time t. So the temperature at x at time t should be a weighted average of the temperature of $f(\phi(0))$ over all paths ϕ which end at x. The shorter the chain is, the more likely it is to affect the point x: because longer chains have higher energy, this explains the presence of the term $e^{-S(\phi)}$.

1.3 Generalizing to mathematical axioms

Even though the path integral for quantum mechanics isn't rigorously defined, we want to use the properties it has (or the path integral for the heat operator, since that's easier) to come up with a set of desired properties we want our mathematical definition of field theories to satisfy. One thing we want to include is higher-dimensional theories than that of a single point (0-dimensional). The following table highlights the motivation for the mathematical axioms of TFT.

physics inspiration	mathematical axioms
(for $d=1$: $\psi : X = \text{map}(pt, X) \to \mathbb{C}$) $(d-1)$-dimensional manifold $Y \rightsquigarrow E(Y) := L^2(map(Y,X))$, the Hilbert space of states on Y	$(d-1)$-dimensional, closed, oriented manifold $Y \rightsquigarrow E(Y)$=complex vector space
(for $d=1$: to interval $[0,t]$, $f_0 \mapsto f_t = e^{-\Delta t} f_0$) d-dimensional bordism $\Sigma : Y_0 \to Y_1, \rightsquigarrow$ linear map $(E(\Sigma)(f_0))(\phi_1) = \int_{\{\Phi:\Sigma\to X \mid \Phi\|_{Y_1}=\phi_1\}} f_0(\Phi\|_{Y_0}) \frac{e^{-S(\phi)}\mathcal{D}\Phi}{Z}$	d-dimensional bordism $\Sigma \rightsquigarrow E(\Sigma) : E(Y_0) \to E(Y_1)$, linear map
Fubini for fiber bundles: $\int_{\{\Phi_{02}:\Sigma_{02}\to X \mid \Phi_{02}\|_{Y_2}=\phi_2\}} = \int_{\{\Phi_{12}:\Sigma_{12}\to X \mid \Phi_{12}\|_{Y_2}=\phi_2\}} \int_{\{\Phi_{01}:\Sigma_{01}\to X \mid \Phi_{01}\|_{Y_1}=\Phi_{12}\|_{Y_1}\}}$	composition of bordisms $\Sigma_{02} = \Sigma_{01} \circ_{Y_1} \Sigma_{12} \rightsquigarrow E(\Sigma_{12}) \circ E(\Sigma_{01})$
for disjoint $Y_0 \sqcup Y_1 \rightsquigarrow L^2(\text{map}(Y_0 \sqcup Y_1, X)) \cong L^2(\text{map}(Y_0, X) \times \text{map}(Y_1, X)) \cong L^2(\text{map}(Y_0, X)) \otimes L^2(\text{map}(Y_1, X))$	$E(Y_0 \sqcup Y_1) \cong E(Y_0) \otimes E(Y_1)$

So what we want of a field theory is something that takes the geometric information of the dynamical structure of space-time and associates to it an algebraic description of the physical states of the system. How to make this precise for general yet realistic quantum field theories is a difficult problem (the geometric version of the bordism category is much more nuanced). But for a basic case, where on the 'geometrical' side we are only concerned with *topological* information, we have the following definition.

Definition 1. A d-dimensional (oriented) topological field theory (d-TFT) is a symmetric monoidal functor

$$E : Bord_d^{\sqcup} \to Vect_{\mathbb{C}}^{\otimes}.$$

Here

- $Bord_d^{\sqcup} = \begin{cases} \text{objects: } (d-1)\text{-dimensional closed, oriented manifolds } Y \\ \text{morphisms:bordism classes:}\{\text{oriented, compact } d\text{-manifolds } \Sigma, \partial\Sigma \cong Y_1 \sqcup Y_0\}/\{\text{diffeo rel } \partial\} \\ \text{composition: gluing bordisms} \\ \text{identity: } id_Y = Y \times [a,b] \\ \text{symmetric monoidal structure: disjoint union, } \sqcup \end{cases}$

- $Vect_{\mathbb{C}}^{\otimes} = \begin{cases} \text{objects: } \mathbb{C}\text{-topological vector spaces} \\ \text{morphisms: continuous, } \mathbb{C}\text{-linear maps} \\ \text{symmetric monoidal structure: (completed) tensor product, } \otimes \end{cases}$

Remark: Different variant of the bordism category give different notions of functorial field theories. For instance, one could equip the bordisms with a Riemannian metric, a conformal structure, a spin structure, a framing, etc.; these would yield Riemannian/conformal/spin/framed/etc. field theories.

1.4 Properties of TFT

In the upcoming lectures, I'll be discussing one of these more specific flavors of functorial field theory, where extra layers of structure will be added to these categories and functors (smoothness, maps to a parameter space, supersymmetry, etc.). But first let's discuss some examples and properties in the basic case of the purely topological field theories.

Proposition 1. Let $E : Bord_d \to Vect$ be a d-TFT. For Σ_1, Σ_2 closed d-manifolds, $E(\Sigma_1 \sqcup \Sigma_2) = E(\Sigma_1) \cdot E(\Sigma_2)$.

Proof. To make sense of the multiplication above, consider the case where Σ_1 is a torus with empty boundary. Since E is a symmetric monoidal functor, it preserves the unit object and we must have $E(\emptyset) = \mathbb{C}$. Therefore $E(\Sigma_1) : \mathbb{C} \to \mathbb{C}$ is just a complex number. Also since E is symmetric monoidal, we must have

$$E(\Sigma_1 \sqcup \Sigma_2) = E(\Sigma_1) \otimes E(\Sigma_2) = E(\Sigma_1) \cdot E(\Sigma_2).$$

□

So TFT's give invariants of manifolds that are exponential. An example of a more explicit TFT is the following:

Example 1. (Euler characteristic) (Stephan's notes, pg. 70-71)

Recall that for M a compact d manifold, the Euler characteristic of M is defined by

$$\chi(M) := \sum_{i=0}^{d} (-1)^i dim H_i(M; \mathbb{Z}/2) \in \mathbb{Z}.$$

Note that the Euler characteristic is additive, i.e.

$$\chi(M_1 \sqcup M_2) = \chi(M_1) + \chi(M_2).$$

We can define a d-TFT as follows: for every closed $(d-1)$-manifold Y, define $E(Y) := \mathbb{C}$. Choose $\lambda \in \mathbb{C}^\times$ and define $E_\lambda(M) := \lambda^{\chi(M)}$. According to the above proposition, we should have $E_\lambda(M_1 \sqcup M_2) = E_\lambda(M_1) \cdot E_\lambda(M_2)$. Indeed, this follows from the additive property of the Euler characteristic.

Proposition 2. Let $E : Bord_d \to Vect_{\mathbb{C}}$ be a TFT. For any closed $Y^{d-1} \in Bord_d$, $E(Y)$ is finite dimensional.

Proof. Set $U := E(Y)$ and $V := E(\overline{Y})$. Consider the bordism $Y \times [0, 1]$. We can view this in two ways: firstly as a bordism from $\overline{Y} \sqcup Y \to \emptyset$, and secondly as a bordism from $\emptyset \to Y \sqcup \overline{Y}$. Composing these two bordisms gives

$$Y \sqcup \emptyset \to Y \sqcup \overline{Y} \sqcup Y \to \emptyset \sqcup Y \cong Y \to Y.$$

Applying the functor E to both sides, we obtain a pairing and copairing. Diffeomorphism invariance implies that the composition is the identity after applying E, so the resulting pairing and copairing are non-degenerate. In particular, this implies that $E(Y)$ is finite dimensional. □

1.5 Exercises

Exercise 1. Let E be a d-TFT. Show that for any object Y in $Bord_d$, $E(Y \times S^1) = \dim_{\mathbb{C}}(E(Y))$.

Exercise 2. Let $E : Bord_d \to Vect$ be a TFT with $\dim E(S^{d-1}) = 1$. Show that $E(S^n) \neq 0$, and for any connected $M : \emptyset \to Y_0, N : Y_1 \to \emptyset$,

$$E(M\#N) = \frac{1}{E(S^n)} E(M) \circ E(N).$$

One interesting question to ask about field theories is to ask for an algebraic description (in terms of generators and relations) for field theories of a given dimension. Looking at the category of field theories of a given dimension (this is a functor category, where the morphisms are natural transformations between the field theories), we can compare these with other algebraic categories. This gives the following nice descriptions for TFT of dimensions 1 and 2.

Exercise 3. The groupoid of 1-TFTs is categorically equivalent to the category of dual pairs $\mathcal{DP}_{\mathbb{C}}$.

(Note: Here we can't just compare 1-TFTs to the category of finite dimensional vector spaces with linear maps, because the latter category is not a groupoid. The category of *dual pairs* is defined to fix that problem.)

Definition 2. The category of *dual pairs*$=\mathcal{DP}_{\Bbbk}$ consists of

- objects: tuples (U, V, b, d); U, V are $\Bbbk$-vector spaces, $b : \Bbbk \to U \otimes V, d : V \otimes U \to \Bbbk$ ("birth" and "death"). These satisfy the Zorro moves: $(d \otimes id_V) \circ (id_V \otimes b) = id_V$, and the analogous thing for id_U. In other words, b, d exhibit the duality of U, V.

- morphisms: $(f, g) : (U, V, b, d) \to (U', V', b', d')$, where $f : U \to U', g : V \to V'$ are linear maps and $d = d' \circ (g \otimes f), b' = b \circ (f \otimes g)$.

Exercise 4. There is an equivalence of groupoids between 2-TFTs and the category of commutative Frobenius algebras.

Definition 3. A Frobenius algebra (A, τ) over $\Bbbk$ is a unital associative algebra A, with a trace map $\tau : A \to \Bbbk$ such that $\tau(xy) = \tau(yx)$ and $A \times A \to \Bbbk, (x, y) \mapsto \tau(xy)$ is non-degenerate.