Mapping Class Groups

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1 Mapping Class Group

Definition 1. Given a compact, orientable surface Σ, the *mapping class group of* Σ is defined to be

$$\mathrm{Mod}(\Sigma) := \mathrm{Diff}^+(\Sigma, \partial\Sigma)/\text{isotopy}.$$

Example 1. The mapping class group of the disk D^2 is trivial.

Example 2. The mapping class group of the torus T^2 is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. To see this, consider the map

$$\psi : \mathrm{Mod}(T^2) \to \mathrm{Aut}(H_1(T^2;\mathbb{Z})) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong GL(2,\mathbb{Z})$$

induced by action on homology. However, this map preserves the algebraic intersection number $\hat{i}$. Using this fact, one can show that $\psi(\mathrm{Mod}(T^2)) \subseteq \mathrm{SL}(2,\mathbb{Z})$. [Exercise: show that this map $\psi : \mathrm{Mod}(T^2) \to \mathrm{SL}(2,\mathbb{Z})$ is an isomorphism.]

To begin talking about the mapping class group, it is helpful to see some nontrivial elements. The "simplest" such nontrivial elements are obtained by twisting about simple closed curves. Let α be a simple closed curve in Σ, and let N be a tubular neighborhood of α. Then N is diffeomorphic to the annlus $A = S^1 \times [0,1]$. Choose an orientation-preserving diffeomorphism $\varphi : A \to N$. Define the twist map $T : A \to A$ by $T(\theta, t) = (\theta - 2\pi t, t)$.

Definition 2. The *Dehn Twist* about a simple closed curve α is the map

$$T_\alpha(x) = \begin{cases} \varphi \circ T \circ \varphi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in \Sigma \setminus N. \end{cases}$$

Remark. If $\alpha \simeq \beta$, then $T_\alpha \simeq T_\beta$. So, it makes sense to talk about the Dehn twist about a homotopy class of curves.

Example 3. Consider the annulus A. One can show that $\mathrm{Mod}(A) \cong \mathbb{Z}$, where the generator is given by the Dehn twist about the core curve of A.

Example 4. Earlier, we saw a map $\psi : \mathrm{Mod}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ given by action on homology. Let's compute this map on some Dehn twists. Consider the two simple closed curves shown below.

[picture of standard simple closed curves on torus]

Dehn twisting these curves about each other, we see

[Dehn twists on torus]

Therefore,

$$T_a \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad T_b \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Recall that these two matrices are the "standard" generators of $\mathrm{SL}(2, \mathbb{Z})$. Therefore, we have shown that $\mathrm{Mod}(T^2)$ is generated by T_a and T_b.

In the these two examples, we saw that the mapping class group is finitely generated by Dehn twists. One can then ask the question: is this always the case? The answer (at least for closed surfaces) was first proven by Dehn in 1930.

Theorem 1. *If Σ is a compact, orientable surface, then $\mathrm{Mod}(\Sigma)$ is finitely generated by Dehn twists.*

In proving this theorem for closed surfaces, Dehn gave a list of $2g(g-1)$ Dehn twists which generate the mapping class group (where g is the genus of the surface). This list was then improved by Lickorish and Humphries who gave collections of size $3g-1$ and $2g+1$, respectively.

[picture of Lickorish twists/Humphries generators]

From here, it is very natural to ask whether the mapping class group is finitely presented. The answer to this question was first proved by McCool in 1975.

Theorem 2. *If Σ is a compact, orientable surface, then $\mathrm{Mod}(\Sigma)$ is finitely presented.*

McCool proved this theorem by giving an algorithm to compute the necessary relations. However, no explicit presentation has been given using this algorithm. To remedy this, Hatcher and Thurston gave a new (topologically motivated) algorithm for computing a finite presentation of $\mathrm{Mod}(\Sigma)$. This algorithm was carried out by Harer to give a concrete finite presentation. Finally, Wajnryb simplified this presentation into 5 "classes" of relations. Some of these are easy to see.

Proposition 1. (Disjointness Relation) If $a \cap b = \emptyset$, then $T_a T_b = T_b T_a$.

Proposition 2. (Braid Relation) If $a \cap b = \{*\}$, then $T_a T_b T_a = T_b T_a T_b$.

2 Symplectic Representation

In one of our first examples, we studied the action of $\mathrm{Mod}(T^2)$ on $H_1(T^2; \mathbb{Z})$ to show that $\mathrm{Mod}(T^2) \cong \mathrm{SL}(2, \mathbb{Z})$. We would like to generalize this situation to an arbitrary compact surface. Let $b \in \{0, 1\}$, and let Σ_g^b be the surface of genus g with b boundary components. Then, just as in the torus case, we have a map

$$\Psi_0 : \mathrm{Mod}(\Sigma_g^b) \to \mathrm{Aut}\left(H_1(\Sigma_g^b; \mathbb{Z})\right) \cong \mathrm{Aut}\left(\mathbb{Z}^{2g}\right) \cong \mathrm{GL}(2g, \mathbb{Z}).$$

Again, we note that elements of $\mathrm{Mod}(\Sigma_g^b)$ preserve the symplectic form $\hat{i}$ (here we are using the fact that $b \in \{0, 1\}$, otherwise $\hat{i}$ is not symplectic). Therefore, $\Psi_0(\mathrm{Mod}(\Sigma_g^b)) \subseteq \mathrm{Sp}(2g, \mathbb{Z})$, and so we get a representation

$$\Psi : \mathrm{Mod}(\Sigma_g^b) \to \mathrm{Sp}(2g, \mathbb{Z}).$$

Now, we have a homomorphism out of $\mathrm{Mod}(\Sigma_g^b)$ and a set of generators of $\mathrm{Mod}(\Sigma_g^b)$ (namely, Dehn twists); how does Ψ act on these generators?

Proposition 3. Let a, b be (isotopy classes of) simple closed curves, then

$$\Psi(T_b)([a]) = [a] + \hat{i}(a, b)[b].$$

In the torus case, we found that this map is an isomorphism (note that $\mathrm{Sp}(2, \mathbb{Z}) \cong \mathrm{SL}(2, \mathbb{Z})$). It turns out that this is not always the case, but Ψ is always at least surjective.

Proposition 4. The map $\Psi : \mathrm{Mod}(\Sigma_g^b) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective.

Outline of Proof. We will use three tools in this proof:

- (i) There is an action of $\mathrm{Sp}(2g, \mathbb{Z})$ on the set of *symplectic bases* for $H_1(\Sigma_g^b; \mathbb{Z})$ which is simply transitive. Here, a symplectic basis is a basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H_1(\Sigma_g^b, \mathbb{Z})$ such that $\hat{i}(a_i, b_j) = \delta_{ij}$ and $\hat{i}(a_i, a_j) = 0 = \hat{i}(b_i, b_j)$ for all i, j.
- (ii) There is an action of $\mathrm{Mod}(\Sigma_g^b)$ on the set of *geometric symplectic bases* which is transitive. Here, a geometric symplectic basis is a collection of oriented simple closed curves $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ in Σ_g^b such that $\{[\alpha_1], [\beta_1], \ldots, [\alpha_g], [\beta_g]\}$ is a geometric symplectic basis for $H_1(\Sigma_g^b; \mathbb{Z})$ and $i(c, d) = \hat{i}([c], [d])$ for all c, d in this basis.
- (iii) If $\{a_1, b_1, \ldots, a_g, b_g\}$ is a symplectic basis for $H_1(\Sigma_g^b; \mathbb{Z})$, then there exists a geometric symplectic basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ with $[\alpha_i] = a_i$ and $[\beta_i] = b_i$ for all i.

From here the proof goes as follows: given any $M \in \mathrm{Sp}(2g, \mathbb{Z})$ and symplectic basis $\mathcal{B}$, apply M to $\mathcal{B}$ to get a new basis $\mathcal{B}'$. Using (iii) realize $\mathcal{B}$ and $\mathcal{B}'$ as two geometric symplectic bases $\mathcal{A}, \mathcal{A}'$. By (ii), there exists an element $f \in \mathrm{Mod}(\Sigma_g^b)$ taking $\mathcal{A}$ to $\mathcal{A}'$. Therefore, $\Psi(f)$ takes $\mathcal{B}$ to $\mathcal{B}'$. Since the action in (i) is *simply* transitive, this implies that $\Psi(f) = M$.

A neat consequence of this proposition is that the Humphries generating set given earlier is "minimal". To see this, we must define a *transvection*.

Definition 3. A transvection in $\mathrm{Sp}(2g, \mathbb{Z})$ is an element whose fixed set (in $\mathbb{R}^{2g}$) has codimension 1.

We note that, by Proposition 3, the image of Dehn twists under Ψ are transvections. Next, we have maps

$$\mathrm{Mod}(\Sigma_g^b) \stackrel{\Psi}{\twoheadrightarrow} \mathrm{Sp}(2g, \mathbb{Z}) \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z}).$$

The final tool required is a proposition of Humphries which says that at least $2g + 1$ transvections are needed to generate $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ (here a transvection of $\mathrm{Sp}(2g, \mathbb{Z}/2\mathbb{Z})$ is the image of a transvection in $\mathrm{Sp}(2g, \mathbb{Z})$ under the map above). Thus, since the two maps above are surjective, we have shown that at least $2g + 1$ Dehn twists are needed to generate $\mathrm{Mod}(\Sigma_g^b)$. So, Humphries' generating set is minimal among generating sets consisting entirely of Dehn twists.

3 Torelli Group

In the previous section, we showed that $\Psi : \mathrm{Mod}(\Sigma_g^b) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective. However, Ψ is not injective in general. In fact, the kernel of Ψ is a heavily studied object known as the *Torelli group*, denoted $\mathcal{I}(\Sigma_g^b)$. From our previous computations, we can compute the Torelli group of several surfaces.

Example 5. If $\Sigma = S^2, D^2, T^2$, then $\mathcal{I}(\Sigma) = 1$. Increasing complexity a bit, we have $\mathcal{I}(\Sigma_1^1) \cong \mathbb{Z}$, where the generator is given by the Dehn twist about the boundary component.

Once we venture a little further though, things get pretty messy, as seen by the following theorem of Mess.

Theorem 3. *The Torelli group of the genus 2 surface is an infinitely generated free group.*

Beyond these examples, there is no simple description for the Torelli group. So what can be said about it? As a result of the Lefshetz fixed point theorem, we get:

Theorem 4. *If $g \geq 0$, $b \in \{0, 1\}$, then $\mathcal{I}(\Sigma_g^b)$ is torsion-free.*

As we did for the mapping class group, it is helpful to describe some "nice" elements of the Torelli group. There are two simple classes of elements:

(a) Separating twists: Dehn twists about separating curves.

(b) Bounding Pair maps (BP maps): a map of the form $T_x T_{x'}^{-1}$, where x and x' are (isotopy classes of) homologous, nonhomotopic, non-nullhomotopic simple closed curves.

Now we can ask, how much of the Torelli group do we get from separating twists and bounding pair maps.

Theorem 5. *(Powell) The Torelli group $\mathcal{I}(\Sigma_g^b)$ is generated by separating twists and bounding pair maps for $g \geq 3$.*

However, there are infintely many separating twists and bounding pair maps. So, is it possible to do any better? As it turns out, Johnson showed that any separating twist is a product of bounding pair maps [Exercise]. Moreover, he showed that any bounding pair map is a product of *genus 1* bounding pair maps; that is, a bounding pair map such that one of the components of the surface cut along the two simple closed curves has genus 1 [Exercise]. Finally, he gave a finite (but gigantic) list of bounding pair maps which generate the Torelli group.

Theorem 6. *For $g \geq 3$, the Torelli group $\mathcal{I}(\Sigma_g^b)$ is finitely generated by bounding pair maps.*

Again, the next obvious question becomes: is $\mathcal{I}(\Sigma_g^b)$ finitely presented? The answer to this question is unknown. So how can we further study the Torelli group? The Torelli group came about when we explored the action of $\mathrm{Mod}(\Sigma_g^b)$ on $H_1(\Sigma_g^b;\mathbb{Z})$; that is, the first nilpotent truncation of $\pi_1(\Sigma_g^b)$. Therefore, to learn more about $\mathcal{I}(\Sigma_g^b)$, it may be helpful to studying the action of $\mathcal{I}(\Sigma_g^b)$ on the *second* nilpotent truncation of $\pi_1(\Sigma_g^b)$. In doing so, we will construct the *Johnson homomorphism*

4 Johnson homomorphism

Let $\mathcal{H} = H_1(\Sigma_g^1;\mathbb{Z})$ and $\pi = \pi_1(\Sigma_g^1, *)$, where $* \in \partial\Sigma_g^1$ (note: here we have restricted to surfaces with one boundary component precisely so that we can choose our basepoint on the boundary). Denote by $\gamma_k(\pi)$ the k-th term in the lower central series for π; that is, $\gamma_1(\pi) = \pi$ and $\gamma_k(\pi) = [\gamma_{k-1}(\pi), \pi]$ for all $k \geq 2$. The goal of this section will be to construct a homomorphism $\tau : \mathcal{I}(\Sigma_g^1) \to \bigwedge^3 \mathcal{H}$.

We have a short exact sequence

$$1 \to \gamma_2(\pi) \to \pi \to \mathcal{H} \to 1.$$

Modding out by $\gamma_3(\pi)$, we get the short exact sequence

$$1 \to \bigwedge^2 \mathcal{H} \to \pi/\gamma_3(\pi) = \Gamma \to \mathcal{H} \to 1.$$

Here, we are using the fact that $\gamma_2(\pi)/\gamma_3(\pi) \cong \bigwedge^2 \mathcal{H}$. Also, note that $\bigwedge^2 \mathcal{H} < \Gamma$ is central.

Next, there is an action of $\mathrm{Mod}(\Sigma_g^1)$ on π which preserves $\gamma_k(\pi)$ (all automorphisms of π do this). Therefore, this action descends to an action of $\mathrm{Mod}(\Sigma_g^1)$ on Γ which preserves $\bigwedge^2 \mathcal{H} < \Gamma$. We claim that the restriction of this action to an action of $\mathcal{I}(\Sigma_g^1)$ on $\bigwedge^2 \mathcal{H}$ is trivial (this requires checking to make sure the action of $\mathrm{Mod}(\Sigma_g^1)$ on $\bigwedge^2 \mathcal{H}$ is what you think it is). Therefore, if $f \in \mathcal{I}(\Sigma_g^1)$ and $x \in \Gamma$, then x and $f(x)$ project to the same element of $\mathcal{H}$. It follows that $x(f(x))^{-1} \in \bigwedge^2 \mathcal{H}$. Therefore, we get a set map $J'_f : \Gamma \to \bigwedge^2 \mathcal{H}$ defined by $J'_f(x) = x(f(x))^{-1}$. Moreover, J'_f is a homomorphism (this relies on the fact that $\bigwedge^2 \mathcal{H}$ is central in Γ), so it factors through the abelianization:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\;J'_f\;} & \bigwedge^2 \mathcal{H} \\ & \searrow \quad \nearrow_{J_f} & \\ & \mathcal{H}. & \end{array}$$

Define the map $\tau_0 : \mathcal{I}(\Sigma_g^1) \to \mathrm{Hom}(\mathcal{H}, \bigwedge^2 \mathcal{H})$ via $\tau(f) \mapsto J_f$. Recall that we wanted a map $\tau : \mathcal{I}(\Sigma_g^1) \to \bigwedge^3 \mathcal{H}$. To get this map, we first note that

$$\mathrm{Hom}\left(\mathcal{H}, \bigwedge^2 \mathcal{H}\right) \cong \mathcal{H}^* \otimes \bigwedge^2 \mathcal{H} \cong \mathcal{H} \otimes \bigwedge^2 \mathcal{H}.$$

Now, there is a canonical embedding of $\bigwedge^3 \mathcal{H}$ into $\mathcal{H} \otimes \bigwedge^2 \mathcal{H}$ given by

$$a \wedge b \wedge c \mapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b),$$

and one can show that the image of τ_0 is contained in the image of this embedding [Exercise]. Therefore, we get our desired map $\tau : \mathcal{I}(\Sigma_g^1) \to \bigwedge^3 \mathcal{H}$ which is called the Johnson homomorphism. An involved computation gives:

Proposition 5.
- If T_x is a separating twist, then $\tau(T_x) = 0$.
- Suppose $T_x T_{x'}^{-1}$ is a bounding pair map. Let Σ' be the component of $\Sigma_g^1 - \{x, x'\}$ which does not contain the boundary of Σ_g^1. Then there exists a subsurface $\Sigma'' \subset \Sigma'$ homeomorphic to Σ_h^1 for some $1 \le h < g$. Let $\{a_1, b_1, \ldots, a_h, b_h\}$ be a symplectic basis for Σ''. Then $\tau(T_x T_{x'}^{-1}) = \pm [x] \wedge (a_1 \wedge b_1 + \cdots + a_h \wedge b_h)$.

Playing around with the second formula, one can show:

Theorem 7. *If $g \ge 2$ then τ is surjective.*

Let $\mathcal{K}(\Sigma_g^1)$ be the subgroup of $\mathcal{I}(\Sigma_g^1)$ generated by separating twists. Since $\bigwedge^3 \mathcal{H}$ is infinite for $g \ge 3$, we get the following corollary.

Corollary 1. If $g \ge 3$, then $\mathcal{K}(\Sigma_g^1)$ has infinite index in $\mathcal{I}(\Sigma_g^1)$.

From Proposition 5, it is clear that $\mathcal{K}(\Sigma_g^1) \subseteq \ker(\tau)$. A deep result of Johnson shows that the reverse inequality is also true.

Theorem 8. *The kernel of τ is generated by separating twists.*

As a final remark, since the theme of this talk has been finiteness properties, we cite a recent result of Church, Ershov, and Putman.

Theorem 9. *If $g \geq 4$, then $\mathcal{K}(\Sigma_g^1)$ is finitely generated.*