Mapping Class Groups Exercises

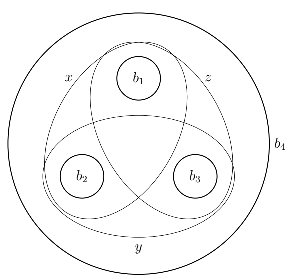
1. Prove that the map $\psi : \mathrm{Mod}(T^2) \to \mathrm{SL}(2, \mathbb{Z})$ is injective.

2. The "Burkhardt generators" for $\mathrm{Sp}(2g, \mathbb{Z})$ ($g \geq 2$) consist of the following elements:

   - Transvection:
   $$(x_1, y_1, x_2, y_2, \ldots) \mapsto (x_1 + y_1, y_1, x_2, y_2, \ldots)$$
   - Factor rotation:
   $$(x_1, y_1, x_2, y_2, \ldots) \mapsto (y_1, -x_1, x_2, y_2, \ldots)$$
   - Factor mix:
   $$(x_1, y_1, x_2, y_2, \ldots) \mapsto (x_1 - y_2, y_1, x_2 - y_1, y_2)$$
   - Factor swap:
   $$\{x_i, y_i\} \leftrightarrow \{x_{i+1}, y_{i+1}\} \text{ for } 1 \leq i \leq g - 1.$$

   Give another proof that $\Psi : \mathrm{Mod}(\Sigma_g^b) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective by finding elements of $\mathrm{Mod}(\Sigma_g^b)$ that map to these generators.

3. Verify the *lantern relation*: given an embedded lantern (diffeomorphic to Σ_0^4) and curves as shown below, where the b_i are the boundary components:



   the Dehn twists about these curves satisfy
   $$T_x T_y T_z = T_{b_1} T_{b_2} T_{b_3} T_{b_4}.$$
   [Hint: it suffices to show this on a collection of curves which fill the lantern.]

4. Use the lantern relation to show that any separating twist in $\mathrm{Mod}(\Sigma_g^b)$ can be written as a product of bounding pair maps for $g \geq 2$.