USING TQFT'S TO SHOW FACTS ABOUT ALGEBRA

HARI RAU-MURTHY

1. Introduction and review of framed bordism

Our goal today is to use the fact that $\pi_1(SO(3)) \cong \mathbb{Z}/2$ to show that for any tensor category $\mathcal{C}$, the quadruple dual $\mathcal{C}^{****} \simeq \mathcal{C}$. We begin with some review.

Definition 1.1. Fix an ambient dimension n. Given any $k \leq n$ and a manifold M of dimension k, a *stable up-to-n framing of M* is a trivialization of $TM^k \oplus \epsilon^{n-k}$.

Example 1.2. Given an embedding $\nu : M^k \hookrightarrow \mathbb{R}^n$, then the *normal bundle* $\nu(f)$ can be used to define a stable up-to-n framing of M^k since $TM \oplus \nu(f) \cong T\mathbb{R}^n$.

Definition 1.3. Suppose we are given a $(k+1)$-dimensional manifold B^{k+1} with boundary $\partial B^{k+1} = M_1^k \sqcup -M_2^k$ where M_1 and M_2 are stable up-to-n framed and $-M_2$ means we invert the normal direction inside $\mathbb{R}^n$. Then B^{k+1} is said to be a *framed bordism* if

1. B has a stable up-to-n framing, and
2. B induces the correct stable up-to-n framings on M_1 and M_2.

We can equip bordism categories with this additional structure.

Definition 1.4. The 2-category $Bord_0^{2-fr}$ has objects disjoint unions of framed points pt_+ and pt_- where $+$ and $-$ refer to the framing being counterclockwise $(+)$ and clockwise $(-)$, 1-morphisms are framed bordisms between these objects, and 2-morphisms are framed bordisms between 1-morphisms rel endpoints.

This 2-category is a monoidal 2-category with monoidal structure given by disjoint union $\sqcup$ and unit $\emptyset$.

We now specify the target category for our TQFT.

Definition 1.5. The target category $\mathcal{C}$ is the 2-category where objects are algebras, 1-morphisms are biomodules ${}_AM_B$, and 2-morphisms are bimodule homomorphisms. This is a monoidal 2-category under tensor product.

Consider symmetric monoidal functors in $Fun^{\otimes}(Bord_0^{2-fr}, \mathcal{C}^{\otimes})$. Note that $F(pt_+)$ is dual to $F(pt_-)$ in the following sense. There are maps

$$F(\emptyset) \to F(pt_+) \otimes F(pt_-),$$

$$F(pt_-) \otimes F(pt_+) \to F(\emptyset)$$

corresponding to the usual cup/U-shaped bordisms, and the composite of these corresponding to the S-shaped bordism is equivalent to the composite corresponding to the trivial bordism by "straightening out" the S-shaped bordism. Therefore we have a map

$$Fun^{\otimes}(Bord_0^{2-fr}, \mathcal{C}^{\otimes}) \to core(\text{dualizable objects of } \mathcal{C}^{\otimes}) =: X$$

where $core(-)$ takes the subcategory of invertible morphisms. This is an ∞-groupoid and we may regard both sides as sufficiently structured (e.g. topological categories, ∞-categories).

Exercise 1.6. *Show that this map is an equivalence.*

2. How does $O(2)$ come in?

There is an $O(2)$-action on both ides of the map above, e.g. we have maps

$$O(2) \to Aut(X)$$

where $Aut(X)$ is the category of endofunctors of X. In fact, the map can be made $O(2)$-equivariant (in the appropriate sense). An element $\gamma \in \pi_1(O(2))$ is sent to a path from the identity functor Id_X to itself, i.e. a natural transformation $Id_X \Rightarrow Id_X$. In the bordism category, if we apply γ to the trivial framed bordism it inserts a loop/curly-Q; we'll call this bordism Q.

Say that $1 \in \pi_1(O(2))$ maps to a natural transformation S. Let's compute $S_A = F(Q)$. We can do this by analyzing each piece of Q. The evaluation map ev coming from the left elbow gives a map $F(pt_+) \otimes F(pt_-) \to F(\emptyset)$ and the coevaluation map coming from the right elbow ev_L gives a map $F(\emptyset) \to F(-(pt_- \sqcup pt_+)) \simeq F(pt_+) \otimes F(pt_-)$.

Exercise 2.1. *Show that ev_L is left-adjoint to ev in the sense of higher category theory (as defined in Tim's talk). To recall, if $f : c \to d$ and $g : d \to c$ are 1-morphisms, then f is left-adjoint to g if there are 2-morphisms $\eta : Id_C \Rightarrow fg$ and $\epsilon : gf \Rightarrow Id_d$ such that*

$$(f \Rightarrow id_c \circ f \overset{\eta \circ id}{\Rightarrow} fg \circ f \overset{id \circ \epsilon}{\Rightarrow} f) = Id_f$$

as 2-morphisms.

Example 2.2. Let $\mathcal{C} = Vect$. Let $V \in Vect$ and let V be dual to V^* under $\otimes$. Then $Hom(V, -) \in BVect_1$ is adjoint to $- \otimes V$ and $Hom(V, -) = \otimes V^*$. In particular, the dual to V is V^*.

We now want to evaluat $F(Q)_A$ where $A \in \mathcal{C}_0$ is an algebra.

Exercise 2.3. *Prove the following:*

1. *The dual to A is A^{op}.*
2. *The evaluation is given by ${}_{A^{op} \otimes A} A_k$.*
3. *The left-adjoint to evaluation is ${}_k Hom(A, k)_{A^{op} \otimes_k A}$.*

Compose to show that

$$S_A = {}_A Hom(A, k)_A.$$

3. 3-categories

Unfortunately, this doesn't buy us enough to achieve the goal we set out with. However, if we pass to the 3-categorical setting, we can proceed as follows.

Definition 3.1. Define a 3-category by setting its objects to be tensor categories, its 1-morphisms to be bimodules ${}_{\mathcal{C}} M_{\mathcal{C}'}$, its 2-morphisms to be functors, and its 3-morphism to be natural transformations.

Example 3.2. Consider the category of modules over $\mathbb{C}[G]$ where G is a group, denoted $Mod_{\mathbb{C}[G]}$. Given G-representations V and W, we can form $V \otimes W$ which is a G-representation by letting G act diagonally. This gives a module structure to $V \otimes W$.

Then we can think of the bordism Q as stable up-to-3 framed, and

$$\pi_1(O(3)) \ni 1 \mapsto F(Q)_{\mathcal{C}} = {}_{\mathcal{C}} \mathcal{C}_{\mathcal{C}^{**}}.$$

Since $2 \cdot 1 = 0 \in \mathbb{Z}/2$, we see that

$${}_{\mathcal{C}} \mathcal{C}_{\mathcal{C}^{**}} \otimes_{\mathcal{C}^{**}} \mathcal{C}_{\mathcal{C}^{****}} \cong_{\mathcal{C}} \mathcal{C}_{\mathcal{C}^{****}} \cong_{\mathcal{C}} \mathcal{C}_{\mathcal{C}}.$$

This implies that the action on $\mathcal{C}^{****}$ on $\mathcal{C}$ is the same as the action of $\mathcal{C}$ on $\mathcal{C}$, and this implies that

$$\mathcal{C}^{****} \cong \mathcal{C}.$$

Exercise 3.3. *Work out the details and technicalities of the last section.*