Talk

Monday, April 23, 2018 8:04 PM

Set-up: The Cobordism Hypothesis

Recall the symm. monoidal (∞, n) category Bord_n^{fr}:

objects = n-framed 0-mflds
1-morphisms = n-framed 1-mflds as bordisms
⋮
n-morphism = n-framed n-mflds
higher : Space of diffeomorphism of n-mfld

"The ∞ stuff comes in when we chose a target category that isn't discrete past n, which it won't today"

Let $\mathcal{C}$ be any other symm. mon. (∞,n)-cat.

Ex $n=2$, $\mathcal{C} = \{\mathrm{Vect}, \text{linear maps}\}$
$n=3$, $\mathcal{C} = \{\mathrm{Alg}, \text{bimod}, \text{bimod maps}\}$

Thm (Cobordism Hypothesis)

There is an equivalence of ∞-grpds

$$\mathrm{fun}^{\otimes}(\mathrm{Bord}_n^{fr}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Core}(\mathcal{C}^{fd})$$

$$f \longmapsto f(pt_+)$$

where $\mathcal{C}^{f.d.}$ = "fully dualizable" i.e. has all duals & adjoints
$\mathrm{Core}(-)$ = throw out non-invertible morphisms.

$O(n)$ action.

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There is an $O(n)$ "action" on Bord_n^{fr}, which induces one on $\mathrm{Core}(\mathcal{C}^{fd})$ [which Mari introduced last time].

What does $G \curvearrowright$ ∞-grpd mean?

"Morphism" $\underset{\text{grp}}{G} \longrightarrow \underset{\text{grpd}}{\mathrm{Aut}(X)}$

<u>Option 1:</u> pick a model for grpds as spaces
$\to$ only defined up to htpy

(<u>Option 2:</u> pick a model for $O(n)$ as a category
$\to$ up to equivalence)

<u>Option 1:</u> We can resolve what a map of topological groups means by applying B

$$\begin{array}{c} \text{"TopGrp"}(G, \mathrm{Aut}(X))/\sim \\ \downarrow B \quad\quad \uparrow \Omega \\ \mathrm{Top}(BG, B\mathrm{Aut}(X))/\sim \end{array}$$

[Really htpy coherent grp - A_∞]

Can Also pass to htpy grps

$O(1)$

$O(1) = \mathbb{Z}/2$ acts on Bord_1^{fr}:

Bord_1^{fr} gen by $\bullet^+$, $\bullet^-$, $\supset$, $\subset$

$\mathbb{Z}/2$ $\quad\quad\quad$ $\mathbb{Z}/2$

We can model $(\mathrm{Bord}_1^{fr})_{h\mathbb{Z}/2}$ "$=$ $(\mathrm{Bord}_1^{fr} \times E\mathbb{Z}/2)_{\mathbb{Z}/2}$"

by choosing a model for $E\mathbb{Z}/2 =$ id  id
& working out product.

Get something equivalent to Bord_1^{un}

A coherent diagram
$$\begin{tikzcd} \mathrm{Bord}_1^{fr} \arrow[r] \arrow[d, "F"'] & \mathrm{Bord}_1^{un} \arrow[dl] \\ \mathcal{C} & \end{tikzcd}$$
is a choice of iso $F(pt_+) \to F(pt_-)$

$\underline{O(2)} = SO(2) \rtimes \mathbb{Z}/2 = S^1 \rtimes \mathbb{Z}/2$

[Know how the $\mathbb{Z}/2$ acts: sends thing $\rightsquigarrow$ dual
[So focus on $SO(2)$.

Fix $\mathcal{C} = (\infty,2)$-cat sym mon.
$X = \mathrm{Core}(\mathcal{C}^{fd})$

"Top Grp map" $\underset{\substack{\uparrow \\ \text{top space}}}{SO(2)} \longrightarrow \underset{\substack{\uparrow \\ \infty\text{-grpd}}}{\mathrm{Aut}(X)} \leftarrow$ has π_i, $i \leq 2$

Convert both to $\underline{\text{connected spaces}}$
Can recover by Ω.

has π_i, $i \leq 3$,

$$\begin{tikzcd} & BSO(2) \arrow[r] & B\mathrm{Aut}(X) \\ & \mathbb{CP}^\infty \arrow[u, phantom, "\shortparallel"] & \\ & \vdots \arrow[u, phantom, "\cup"] & \\ & \mathbb{CP}^3 \arrow[u, phantom, "\cup"] & \\ D^4 \arrow[r] & \mathbb{CP}^2 \arrow[u, phantom, "\cup"] \arrow[uuuur] & \\ S^3 \arrow[u, "\iota"] \arrow[r, "\eta"'] & S^2 \arrow[u, phantom, "\cup"] \arrow[uuuuur, bend right, "\alpha"'] & \end{tikzcd}$$

[B is well def up to htpy
but we know a model
from nerve/realization

$X = \mathrm{Cat}$ 2-cat
$\mathrm{Aut}(X) =$

$[\alpha] \in \pi_2 B\mathrm{Aut}(X) = \pi_1 \mathrm{Aut}\, X$, $\mathrm{Aut}\, X =$
· functors $X \longrightarrow X$
· squares $X \to X$
· Δ^2's $X \to X$

$[\alpha]$ is a "loop" at "id": ie
$S : \mathrm{id}_X \longrightarrow \mathrm{id}_X$
$S(A) : A \longrightarrow A \quad \forall A \in X$
natural in A.

Claim an $SO(2)$ action on X up to htpy is exactly

$$S : Id_X \to Id_X \in \pi_2 BAut(X) = \pi_1(Aut X)$$

s.t.

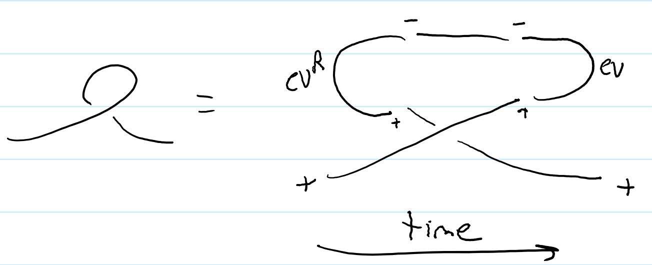
$$\eta S \in \pi_3 BAut(X) = \pi_2 Aut(X)$$

is null

"In diagram, we only add higher cells above $\mathbb{C}P^2$ - same 3-type"

Claim: S_A is ${}_A\mathrm{Hom}(A,k)_A$ in $Alg_{\mathbb{C}}$ as a 2-cat

pf We saw this last time/exercises



& computing duals & adjoints

$$\left(\begin{tikzcd} A\otimes 1 \arrow[r, "ev^R"] & A\otimes A\otimes A^\vee \arrow[r, "ev"] & A \end{tikzcd} \right)$$

Suppose some functor $Bord_2^{fr} \to Alg_{\mathbb{C}}$ "is" a fixed pt for $SO(2)$

$$pt_{+1} \mapsto A$$

Then: $\pi_0 \Rightarrow$ nothing

$\pi_1 \Rightarrow A \cong \mathrm{Hom}(A,k)$ as (A,A)-bimodules Given by some iso $\varphi : A \to \mathrm{Hom}(A,k)$

$\pi_2 \Rightarrow$ nothing

Then we define $\lambda(a) = \varphi(1)(a) = (a\cdot\varphi(1))(1) = \varphi(a)(1)$

$\Rightarrow A$ is a symmetric frob alg

Note Here, an extension →

$$\begin{tikzcd} \mathrm{Bord}_2^{fr} \arrow[r] \arrow[d] & \mathrm{Alg}_{\mathbb{C}} \\ \mathrm{Bord}_2^{or} \arrow[ur, dashed] & \end{tikzcd}$$

exists if ${}_A A_A \simeq {}_A \mathrm{Hom}(A, \mathbb{C})_A$ abstractly

but there can be multiple extensions, corresponding to choice of extension

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Modeling η

Whitehead's Certain Exact Seq, X simply-connected & $\pi_{3,4}X = 0$

$$\begin{tikzcd} 0 \arrow[r] & H_4(X) \arrow[r] & \Gamma(\pi_2(X)) \arrow[r, "q"] & \pi_3(X) \arrow[r] & H_3(X) \arrow[r] & 0 \\ & & \pi_2(X) \arrow[u] \arrow[ur, "\eta"'] & & & \end{tikzcd}$$

Can get q from Postnikov tower for X

$$\begin{tikzcd} K(\pi_3(X), 3) \arrow[r] & X \arrow[d] \\ & K(\pi_2(X), 2) \end{tikzcd}$$

Can produce a map $k_2: K(\pi_2(X), 2) \to K(\pi_3(X), 4)$
$\rightsquigarrow q$ in terms of that

O(3)

	$SO(2)$	$SO(3) \simeq \mathbb{RP}^3$
π_1	$\mathbb{Z}$	$\mathbb{Z}/2$
π_2	0	0
π_3	0	$\mathbb{Z}$

$\longrightarrow \Longrightarrow S^2 \simeq \Sigma \mathrm{Id}: \mathcal{C} \to \mathcal{C}$

[Hari told us about this last time]

Claim: an $SO(3)$ action on a sym. mon. 3-cat $\mathcal{C}$

$$\begin{array}{lll} S: Id_{\mathcal{C}} \longrightarrow Id_{\mathcal{C}} & \text{equivalence} & \\ \sigma: q^{(S)} \longrightarrow Id_{Id_{Id_{\mathcal{C}}}} & \text{equivalence} & \text{Surprising} \\ R: S \circ S \longrightarrow Id_{Id_{\mathcal{C}}} & \text{eq.} & \end{array}$$

s.t. $\quad \mathbb{Z}\frac{a}{2}(R, S) = 0$

fact in a 3-category, adjoints of 1-morphisms are two sided when they & a few other things exist

Examples of TFTs

$G =$ finite grp

$A = Map(G, \mathbb{C}) \rightsquigarrow (f_1 f_2)(x) = \sum_{ab=x} f_1(a) f_2(b)$

$\implies \exists$ TFT $F: Bord_2^{fr} \longrightarrow Alg_{\mathbb{C}}$
s.t. $\quad pt_+ \longmapsto A$

Can compute $F(S^1) = A \otimes_{A \otimes A^{op}} A$

Extend to $Bord_2^{or}$ via $tr: Map(G, \mathbb{C}) \longrightarrow \mathbb{C}$

$\rightsquigarrow A \otimes_{A \otimes A^{op}} A \simeq$ Class fns on $G \cong \mathbb{C} \otimes$ Rep. Ring.

Can explicitly construct F via Morse theory
or $F(M)$ in terms of G-bundles on M
$\rightsquigarrow$ finite version of path-integral, but IDK much

$\mathcal{C} = \{$tensor categories$\} \Rightarrow$ "3-cat analog of $Alg_{\mathbb{C}}$"

over $\mathbb{C}$

Just like Alg to Vect

Ex: Category of $\mathbb{C}$-reps of G