Exercises

Sunday, April 29, 2018 6:53 PM

I. Whitehead's certain exact sequence

X = space s.t. $\pi_1(X) = \pi_i(X) = 0$ $i > 3$

Let $SP(X) = \infty$ sym product: a space s.t. $\pi_i(SP(X)) = \tilde{H}_i(X)$ &

$\exists$ map

$$X \longrightarrow SP(X)$$

s.t.

$$\pi_i(X) \longrightarrow \tilde{H}_i(X)$$

is Hurewicz.

Let $F = \text{fiber}(X \longrightarrow SP(X))$, then LES in π_* has form

$$0 \longrightarrow H_4(X) \longrightarrow \pi_3(F) \longrightarrow \pi_3(X) \longrightarrow H_3(X) \longrightarrow 0$$

<u>Def</u> for A an abelian grp, let $\Gamma(A)$ be the grp such that

$$\mathrm{Hom}^{\text{quadratic}}(A, B) = \mathrm{Hom}_{Ab}(\Gamma(A), B).$$

Here, a map $f: A \to B$ is <u>quadratic</u> if

- $f(-a) = f(a)$
- $f(a) + f(b) + f(c) + f(a+b+c) = f(a+b) + f(b+c) + f(c+a)$

<u>1)</u> Show $\pi_3(F) = \Gamma(\pi_2(X))$

<u>Rmk</u> Such an f has $f(na) = n^2 f(a)$ and
$f(a+b) - f(a) - f(b)$ is a sym. bilinear form.

2) Let

$$\begin{tikzcd} K(\pi_3 X, 3) \arrow[r] & X \arrow[d] \\ & K(\pi_2(X), 2) \end{tikzcd}$$

be the Postnikov tower of X

& let $k_2 : K(\pi_2(X), 2) \to K(\pi_3(X), 4)$
be the continuation of the fiber sequence.
Using universal coefficients + $K(A,n)$ representing $H^n(-;A)$, show that, in an appropriate sense,

$$k_2 = q : \Gamma(\pi_2(X)) \to \pi_3(X)$$

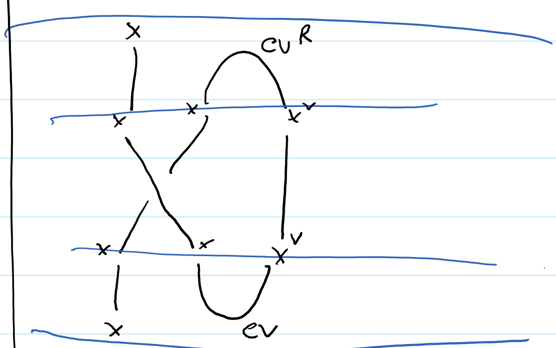
II. Serre Automorphism

Let S be the natural transformation

$$S : \mathrm{Id}_{\mathrm{Bord}_n^{fr}} \to \mathrm{Id}_{\mathrm{Bord}_n^{fr}}, \; n \geq 2$$

Notation:

- $X \in (\mathrm{Bord}_n^{fr})_0$
  e.g. $X = pt_+$
- ev^R = right adjoint to ev
  & ev^L similar
- $X^\vee$ = dual of X (opposite orientation)



$$\begin{aligned}
& X \amalg \emptyset \\
& \downarrow \mathrm{id} \amalg ev^R \\
& X \amalg X \amalg X^\vee \\
& \downarrow \mathrm{swap} \amalg \mathrm{id} \\
& X \amalg X \amalg X^\vee \\
& \downarrow \mathrm{id} \amalg ev \\
& X \amalg \emptyset
\end{aligned}$$

Then S induces a natural transformation

$$S : \mathrm{Id}_{\mathrm{Fun}(\mathrm{Bord}_n^{fr}, \mathcal{C})} \to \mathrm{Id}_{\mathrm{Fun}(\mathrm{Bord}_n^{fr}, \mathcal{C})}$$

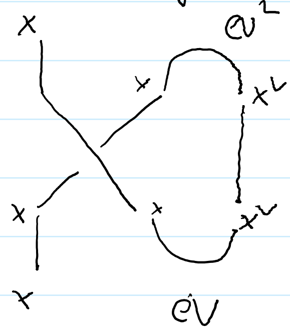
for each symmetric monoidal (∞, n) category $\mathcal{C}$
Suppose WLOG that $\mathcal{C}$ is fully dualizable. Then the cobordism hypothesis induces a nat. trans

$$S : \mathrm{Id}_{\mathrm{Core}(\mathcal{C})} \to \mathrm{Id}_{\mathrm{Core}(\mathcal{C})}$$

3) Show that $S_X : X \to X$ is an isomorphism for each object $X \in \mathrm{Bord}_n^{fr}$ with inverse

$X,$ ev^L

For each object $x \in \mathrm{Bord}_n^{fr}$ with inverse



4) Let $n \geq 3$.

Show that $S^2 : \mathrm{Id}_{Core(e)} \longrightarrow \mathrm{Id}_{Core(e)}$

is equivalent to

$$\mathrm{Id} : \mathrm{Id}_{Core(e)} \longrightarrow \mathrm{Id}_{Core(e)}$$

I.E. That S is cobordant to S^{-1} for $n \geq 3$.