# TOPOLOGICAL MODULAR FORM THEORY VIA QUANTUM FIELD THEORIES?

STEPHAN STOLZ

## 1. Introduction

This is a report on work with Peter Teichner over the past few years.

## 2. (Topological) modular forms

**Definition 2.1.** Suppose that $\mathfrak{h} \xrightarrow{f} \mathbb{C}$ is a complex-valued function on the upper half plane. It is an *integral weak modular form of degree* $n$ (equivalently weight $n/2$) if

(1) $f$ is holomorphic,
(2) for each $\tau \in \mathfrak{h}$, we have $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{n/2} f(\tau)$ where $a, b, c, d$ are entries in a matrix in $SL_2(\mathbb{Z})$, and
(3) for each $\tau \in \mathfrak{h}$, we have a Laurent series expansion called the *q-expansion* of the form

$$f(\tau) = \sum_{k=-N}^{\infty} a_k q^k,$$

and
(4) (integrality) for all $k$, we have $a_k \in \mathbb{Z}$,

We will denote the group of weak integral modular forms of degree $n$by $MF_n$. Putting all of these together defines a graded ring

$$MF_* := \bigoplus_{n \in \mathbb{Z}} MF_n.$$

**Remark 2.2.** *Note that if we consider the matrix* $(1, 1, 0, 1)$*, then we see that* $f(\tau + 1) = f(\tau)$*. Therefore in particular* $f$ *factors through the quotient*

$$\mathfrak{h}/\mathbb{Z} \xrightarrow{f} \mathbb{C}.$$

*Note that the quotient* $\mathfrak{h}/\mathbb{Z}$ *is holomorphically equivalent to the punctured 2-disk* $\overset{\circ}{D}{}^2$ *via the assignment*

$$[\tau] \mapsto q := e^{2\pi i \tau}.$$

*This motivates condition* (3) *above.*

There are various constructions of topological modular forms. Today we are interested in the *periodic topological modular forms spectrum* $TMF$. There is a map

$$\pi_* TMF \to MF_*$$

which is rationally an isomorphism.

We know the right-hand side explicitly. We have

$$MF_* \cong \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 2^6 \cdot 3^3 \cdot \Delta)$$

where $|c_4| = 8$, $|c_6| = 12$, and $|\Delta| = 24$. After imposing the "weak" condition on modular forms, we obtain

$$MF_* \cong (\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 = 2^6 \cdot 3^3 \cdot \Delta))[\Delta^{-1}]$$

which is periodic with periodicity element $\Delta$.

It turns out that $\pi_* TMF$ is also periodic - it has periodicity element $\Delta^{24}$.

## 3. Connections between physics and topology

Let $M$ be a closed string $n$-manifold, where "string" means we have a trivialization of $TM|_{M^{(4)}}$. In topology, this gives a class $[M] \in \Omega_n^{String} \cong \pi_n MString$. By the work of Ando-Hopkins-Rezk, there is a string orientation map

$$\pi_n MString \to \pi_n TMF.$$

We then have the map above

$$\pi_n TMF \to MF_n.$$

If one traces $[M]$ all the way through these, we obtain a modular form $W(M) \in MF_n$ called the *Witten genus*.

On the other hand, we can define the Witten genus using physics, although possibly less rigorously. Witten conjectured the existence of a field theory called the *nonlinear $\sigma$-model of* $M$ called $\sigma_M$. This is supposed to be an element in the space $2|1 - EFT_n$, i.e. it is a degree $n$ Eucliden field theory of dimension $2|1$. It turns out that in this case, we only care about the connected component of the space $\sigma_M$ lies in. Therefore we can reduce to studying $\pi_0$. The *partition function*

$$\pi_0(2|1 - EFT_n) \xrightarrow{Z} MF_n$$

gives a map to $MF_n$, and the claim is that $Z(\sigma_M) = W(M)$.

In diagramatic form, we have

$$\begin{array}{ccccc} M & \longrightarrow & (M) & & \\ \downarrow & & & \searrow & \\ (\sigma_M) & \xrightarrow{Z} & Z(\sigma_M) & \xrightarrow{=} & W(M) \end{array}$$

What does this suggest about the relationship between $TMF$ and field theories? Note that $\pi_n(TMF)$ may be thought of as $\pi_0(TMF_n)$ where $TMF_n$ is the $n$-th space in the $\Omega$-spectrum $TMF$. This motivates the following conjecture:

**Conjecture 3.1.**

$$TMF_n \simeq 2|1 - EFT_n.$$

## 4. Field theories

**Definition 4.1.** A *Euclidean field theory of dimension* 2 *and degree* 0, abbreviated 2-EFT, is a symmetric monoidal functor

$$2 - EBord \to Vect$$

where the left-hand side is the bordism category of Euclidean 2-manifolds and the right-hand side is the category of topological vector spaces.

**Remark 4.2.** *By "Euclidean 2-manifold," we mean oriented 2-manifolds equipped with a flat Riemannian metric. Equivalently, we can view these as manifolds equipped with the* rigid geometry *given by $(\mathbb{R}^2, \mathbb{R}^2 \rtimes SO(2))$ where $\mathbb{R}^2$ on the first part is the model space and $\mathbb{R}^2 \rtimes SO(2)$ is the group where transition functions come from, where $\mathbb{R}^2$ corresponds to translation and $SO(2)$ corresponds to rotation.*

**Definition 4.3.** Let $E$ be a 2-EFT. Note that we can only evaluate on flat torii $T_{\ell,\tau}$ obtained as quotients

$$T_{\ell,\tau} \cong \mathfrak{h}/\mathbb{Z}\{\ell\tau, \ell\}$$

where $\ell \in \mathbb{R}_{>0}$, $\tau \in \mathfrak{h}$. The *partition function of* $E$ is defined by

$$Z_E : \mathbb{R}_{>0} \times \mathfrak{h} \to \mathbb{C},$$

$$(\ell, \tau) \mapsto E(T_{\ell,\tau}).$$

The partition function can be shown to be smooth. Further, we know that $SL_2(\mathbb{Z})$ acts on $\mathfrak{h}$. In fact, we can produce an action

$$SL_2(\mathbb{Z}) \times (\mathbb{R}_{>0} \times \mathfrak{h}) \to \mathbb{R}_{>0} \times \mathfrak{h},$$
$$(g, (\ell, \tau)) \mapsto (|c\tau + d|\ell, \frac{a\tau + b}{c\tau + d}).$$

One can show that there is an isometry

$$T_{g(\ell,\tau)} \cong T_{(\ell,\tau)}.$$

Therefore the partition function $Z_E$ is $SL_2(\mathbb{Z})$-invariant.

**Definition 4.4.** A $2|1$-EFT $\hat{E}$ is a symmetric monoidal functor

$$\hat{E} : 2|1 - EBord \to Vect$$

where the left-hand side is the category of Euclidean $2|1$-manifolds and $Vect$ is the category of $\mathbb{Z}/2$-graded topological vector spaces.

**Remark 4.5.** *By "Euclidean $2|1$-manifold,", we mean manifolds modeled on $(\mathbb{R}^{2|1}, \mathbb{R}^{2|1} \rtimes Spin(2))$. Note that the translation part $\mathbb{R}^{2|1}$ is not abelian - this produces some interesting phenomena.*

The composite

$$2 - EBord_{Spin} \xrightarrow{- \times \mathbb{R}^{0|1}} 2|1 - EBord \xrightarrow{\hat{E}} Vect$$

defines a 2-EFT $E$ associated to $\hat{E}$. We can then define

$$Z_{\hat{E}} := Z_E \in C^\infty(\mathbb{R}_{>0} \times \mathfrak{h}).$$

**Theorem 4.6.** *Let $\hat{E}$ be a $2|1$-EFT. Then*
(1) $Z_{\hat{E}}(\ell, \tau)$ *is* independent *of $\ell$, and*
(2) $Z_{\hat{E}} \in MF_0$.

This defines a map

$$\pi_0(2|1 - EFT) \to \pi_0(TMF)$$

which is the $n = 0$ case of what we wanted!

*Outline of proof.* We want to understand

$$2 - EBord \xrightarrow{E} Vect.$$

An interesting object of the left-hand side is the circle $S^1_\ell$ of length $\ell$. Suppose this maps to $V := E(S^1_\ell)$. What are the interesting morphisms? These are cylinders obtained by taking the quotient

$$C_{\ell,\tau} := \mathfrak{h}/\mathbb{Z}\{\ell\tau\},$$

i.e. bordisms from $S^1_\ell$ to itself. This maps to $E(C_{\ell,\tau}) \in End(V)$, and allows us to define a map

$$\mathfrak{h} \to End(V),$$
$$\tau \mapsto E(C_{\ell,\tau})$$

which turns out to be a homomorphism from the abelian semigroup $\mathfrak{h}$ to the group $End(V)$. We can then think of the left-hand side $\mathfrak{h}$ as the moduli space of cylinders. There is an inclusion

$$\mathfrak{h} \subseteq \hat{\mathfrak{h}}$$

from $\mathfrak{h}$ into the moduli space of $2|1$-cylinders, and this is compatible with the homomorphism

$$\hat{\mathfrak{h}} \to End(V).$$

Moreover, both of these maps are smooth. Therefore we can differentiate to obtain a diagram of Lie algebras

$$\begin{array}{ccc} Lie(\mathfrak{h}) & \longrightarrow & End(V) \\ \big\downarrow \subseteq & \nearrow & \\ Lie(\hat{\mathfrak{h}}) & & \end{array}$$

Let $\frac{\partial}{\partial z} \mapsto L_0$ and $\frac{\partial}{\partial \bar{z}} \mapsto \bar{L_0}$ be the images of the generators under the top horizontal map. We have

$$Lie(\hat{\mathfrak{h}}) = Lie(\mathfrak{h}) \oplus \mathbb{C}Q \cong Lie(\mathbb{R}^{2|1})$$

with bracket

$$\frac{1}{2}[Q, Q] = Q^2 = \frac{\partial}{\partial \bar{z}}.$$

The key point, then, is that $\frac{\partial}{\partial \bar{z}}$ becomes the square-root of an element. Therefore if we say that

$$Q \mapsto \bar{G_0},$$

then we obtain the relation $\bar{G_0}^2 = \bar{L_0}$.

Finally, we want to calculate

$$Z_{\hat{E}}(\ell, \tau) = Z_E(\ell, \tau) = E(T_{\ell,\tau}) \in \mathbb{C}.$$

Above, we were calculating $E(C_{\ell,\tau}) \in End(V)$. What is the relationship? If we glue the ends of the cylinder together we obtain a torus; algebraically, this corresponds to the (super)trace and we obtain

$$E(T_{\ell,\tau}) = str(E(C_{\ell,\tau}).$$

We then find that

$$V = \bigoplus V_{a,b}$$

where the sum runs over $a \in Spec(L_0)$ and $b \in Spec(\bar{L_0})$. Thus we see that

$$Z_{\hat{E}}(\ell, \tau) = \sum str(E(C_{\ell,\tau})|_{V_{a,b}}).$$

The action of $L_0$ and $\bar{L_0}$ are given by $q^a$ and $\bar{q}^b$. We then have

$$\sum str(E(C_{\ell,\tau})|_{V_{a,b}}) = \sum q^a \cdot \bar{q}^b \cdot sdim(V_{a,b})$$

where the superdimension $sdim(V_{a,b}) = dim(V^+_{a,b}) - dim(V^-_{a,b})$ is zero for $b \neq 0$ - this follows from observing that we have

$$\bar{G}_0 : V^+_{a,b} \to V^-_{a,b}$$

and this is an isomorphism, which can be seen by applying $\bar{G}_0$ and $\bar{L}_0$ in appropriately. We conclude that

$$Z_{\hat{E}}(\ell, \tau) = \sum_{a \in Spec(L_0)} q^a \cdot sdim V_{a,0}.$$

Further, $a \in \mathbb{Z}$. This expression then implies that we have an integral modular form. □