Notre Dame Graduate Student Topology Seminar, Spring 2018

Lecture 2: Supersymmetric field theories

2.1 The goal

The goal of the rest of this minicourse it to begin to describe a result from Stolz and Teichner. They have the following classification:

Theorem 1. [3], [2] For smooth manifolds X, one has the natural group isomorphisms:

$$0|1\text{-EFT}^n(X) \cong \begin{cases} \Omega^{ev}_{cl}(X), & n \text{ even} \\ \Omega^{odd}_{cl}(X), & n \text{ odd} \end{cases}$$

When you pass to concordance classes, you get the following isomorphisms:

$$0|1\text{-EFT}^n[X] \cong \begin{cases} H^{ev}_{dR}(X), & n \text{ even} \\ H^{odd}_{dR}(X), & n \text{ odd} \end{cases}$$

Conjecture 1. It is known that the space of $1|1$-EFT's is homotopy equivalent to $BO \times \mathbb{Z}$, which leads to the following conjecture:

$$1|1\text{-EFT}^n[X] \cong KO^n(X)$$

Conjecture 2. [3] There is an isomorphism $2|1\text{-EFT}^n[X] \cong \text{TMF}^n(X)$, compatible with the multiplicative structure.

Definition 1. Two field theories $E_0, E_1 \in d|\delta\text{-EFT}^n(X)$ are *concordant* if there exists a field theory $E' \in d|\delta\text{-EFT}^n(X \times \mathbb{R})$ and an $\epsilon > 0$ such that $E' \cong p_1^*(E_0)$ on $X \times (-\infty, \epsilon)$ and $E' \cong p_1^*(E_1)$ on $X \times (1-\epsilon, \infty)$.

Note: passing to concordance classes forgets geometric information while remembering topological information (for example, two closed n-forms are concordant iff they represent the same de Rham cohomology class; two vector bundles with connection are concordant iff they are isomorphic).

Stolz and Teichner have been working on this project for many years, so I don't intend to go into the more complicated details of it. (Perhaps someone else can speak about the formal group relationships between $H_{dR}(X), K(X)$ and $TMF(X)$ in one of the later talks.) For this minicourse, I'm just going to focus on the $0|1$-dimensional case. I'm also just going to consider the topological version, not the Euclidean one (perhaps how the geometry is incorporated into the bordism category by using collars could also be a topic for a future talk). But I think even this simple case will be helpful for illustrating how these twisted, supersymmetric field theories work. For the topological version, we have the following classification:

Theorem 2. For $X \in \text{Man}$, there are natural group isomorphisms, compatible with multiplication:

$$0|1\text{-TFT}^n(X) \cong \Omega^n_{cl}(X)$$

Taking concordance classes, one gets the following isomorphisms:

$$0|1\text{-TFT}^n[X] \cong H^n_{dR}(X)$$

Over the next few weeks, my goal is to unpack this most basic case of the theorem: the relationship between $0|1$-TFT$[X]$ and de Rham cohomology. Even though in this simple case some of the extra structure might seem more complicatedly phrased than necessary, I want to describe these structures of supersymmetric, twisted field theories in general, such that they apply to the higher dimensional cases.

There are several additional levels of complexity that we will need to add to the basic definition of a functorial field theory that we discussed last time:

- supersymmetry
- field theories where the categories are family versions of the ones we discussed before
- field theories over a manifold X
- twisted field theories/field theories of a higher degree

To build up to describing this theorem, I'm going to start today by discussing the supersymmetric part: requiring the manifolds in the bordism category to be supermanifolds. In physics, the motivation for supersymmetric field theories comes from wanting to describe systems that have symmetries between particles of different statistics (e.g. bosons and fermions).

2.2 Supermanifolds

An ordinary manifold M can be described by a pair $(M, \mathcal{O}_M)$, where M is the underlying topological space, and $\mathcal{O}_M$ is the structure sheaf of the manifold (the sheaf of smooth functions). For ordinary manfolds, the structure sheaf is a sheaf of commutative algebras.

In an analogous way, we will define supermanifolds in terms of their sheaf of functions: but in this case the sheaf of functions is a sheaf of commutative *superalgebras*.

Definition 2. A commutative superalgebra $A \in SAlg$ is a $\mathbb{Z}/2$-graded vector space $A = A^{ev} \oplus A^{odd}$, with a multiplication map $m : A \otimes A \to A, a \otimes b \mapsto ab$ and unit map $u : \Bbbk \to A$ such that

$$ab = (-1)^{|a|\cdot|b|} ba$$

where $|a|$ is the parity of a homogeneous element of A, $|a| = \begin{cases} 0, & \text{if } a \in A^{ev} \\ 1, & \text{if } a \in A^{odd} \end{cases}$

Example 1. Any commutative algebra is a commutative superalgebra, with only an even part. For example, $\mathbb{R}[t^1, ..., t^p] \subset C^\infty(\mathbb{R}^p)$.

Example 2. Exterior algebras: $\Lambda^*(\mathbb{R}^q) = \bigoplus_{k=0}^\infty \Lambda^k(\mathbb{R}^q)$
$\Lambda[\theta^1, ..., \theta^q] =$ exterior algebra generated by odd elements, θ^i; $\theta^i \wedge \theta^j = (-1)\theta^j \wedge \theta^i$

Example 3. $\mathbb{R}[t^1, ..., t^p] \otimes \Lambda[\theta^1, ..., \theta^q] \subset C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, ..., \theta^q]$

Example 4. Differential forms: Let $X \in$ Man. One can decompose $\Omega^*(X) = \bigoplus_{k=0}^\infty C^\infty(X, \Lambda^k(T^*X))$ as $\Omega^*(X) = \Omega^{ev}(X) \oplus \Omega^{odd}(X)$. The wedge product give the multiplicative structure for this as a commutative superalgebra.

Definition 3. A **supermanifold** M of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ where:

- $|M|$ is a topological space of dimension p (called the 'reduced manifold')
- $\mathcal{O}_M$ is a sheaf of commutative superalgebras such that

$$(|M|, \mathcal{O}_M) \underset{locally}{\cong} (\mathbb{R}^p, C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, ..., \theta^q]) =: \mathbb{R}^{p|q}$$

Note that because we have partitions of unity, we can just look at the global sections; in what follows I will write $C^\infty(M)$ for $\mathcal{O}_M(M)$.

The morphisms in the category of supermanifolds (SMan) are maps between these ringed spaces. Since working with maps involving sheaves is more messy, we'll use the following proposition as giving a more convenient description of the morphism sets.

Proposition 1. For $M, N \in SMan$,

$$SMan(M, N) := SAlg(C^\infty(N), C^\infty(M))$$

where the morphisms in $SAlg$ are superalgebra homomorphisms which are grading preserving.

Example 5. In the definition of supermanifolds, we already saw the most basic example of one of dimension $p|q$: $\mathbb{R}^{p|q} := (\mathbb{R}^p, C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, ..., \theta^q]$.

Example 6. An ordinary p-manifold $M = (M^p, \mathcal{O}_M) \in Man$ is a supermanifold of purely even dimension, i.e. a supermanifold of dimension $p|0$.

Example 7. Let $E \to X^p$ be a $\mathbb{R}$-vector bundle of rank q. Then one can form the algebra bundle of alternating multilinear forms on E: $\Lambda(E^*) \to X$. Taking the sheaf of sections of this bundle as the structure sheaf gives a supermanifold $(X, C^\infty(X, \Lambda(E^*))) =: \Pi E$, which we call the *parity reversed bundle*. This is a supermanifold of dimension $p|q$.

To see why this is the case, consider the trivial bundle $E = \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p = X$. The structure sheaf of ΠE for this bundle is: $C^\infty(\mathbb{R}^p, \Lambda((\mathbb{R}^q)^*)) = C^\infty(\mathbb{R}^p) \otimes \Lambda[\theta^1, ..., \theta^q]$. Since every vector bundle is locally isomorphic to the trivial bundle, the claim follows.

In fact, every supermanifold is isomorphic to ΠE for some vector bundle E. (This is Batchelor's Theorem.)

Example 8. In particular, let's look at the tangent bundle $TX \to X^p$. Then $\Pi TX = (X, C^\infty(\Pi TX))$, where $C^\infty(\Pi TX) = C^\infty(X, \Lambda T^*X) = \Omega^*(X) \Rightarrow \Pi TX = (X, \Omega^*(X))$ is a supermanifold of dimension $p|p$.

This gives a first relationship between differential forms and supermanifolds; however, differential forms have more structure than simply being a superalgebra–there is a $\mathbb{Z}$-grading and a differential. Do these extra structures appear in the supermanifold picture? To see them, we'll need to look at mapping spaces of supermanifolds.

2.3 Generalized supermanifolds

Ideally, we would like the mapping space to again be a supermanifold. However, this is *not* the case (even in the case of ordinary manifolds, $Man(M,N) \notin Man$ if $\dim(M) \neq 0$).

Our solution will be to pass to a larger category that does have these mapping spaces (i.e. has internal hom's); we will call this the category of *generalized supermanifolds*. We form this category by taking the Yoneda embedding of our category $SMan$ into the category of presheaves on $SMan$:

$$\begin{aligned} SMan &\longrightarrow Fun(SMan^{op}, Set) \\ X &\longmapsto SMan(-,X) =: X(-) \end{aligned}$$

Definition 4. The category of generalized supermanifolds is the functor category

$$\widehat{SMan} := Fun(SMan^{op}, Set).$$

Note that because the Yondeda embedding is fully faithful, $SMan(-,X)$ completely determines X up to isomorphism, and vice versa, so we don't lose information about supermanifolds by passing to the generalized supermanifold category.

Also, note that the category of generalized supermanifolds has internal hom objects: for $Y, Z \in SMan$, define the internal hom object $\underline{SMan}(Y,Z) := SM(- \times Y, Z) \in \widehat{SMan}$.

Another advantage of the category of generalized supermanifolds is the functor of points approach. Because the Yoneda embedding gives us a bijection between:

$$SM(X,Y) \longleftrightarrow \{\Phi : X(-) \Rightarrow Y(-)\}$$

we have a much easier way of comparing two supermanifolds X, Y. If we want to tell whether X and Y are isomorphic, the above bijection tells us that it suffices to look at the natural transformations between the associated generalized supermanifolds. Each natural transformation $\eta : X(-) \Rightarrow Y(-)$ is a collection of maps $\eta_S : X(S) \to Y(S), \forall S \in SMan$. These η_S are maps between sets. So if we want to tell whether $X \cong Y$, instead of comparing sheaves, which is more messy and complicated, the question can be reduced to determining whether set maps between the S-point sets of X and Y are bijections, which is an easier question.

We will use this functor of points approach to give an alternative description of the differential forms $\Omega^*(X)$, which will be relevant to the $0|1$-TFT-classification theorem this minicourse is discussing. From this description we will also see where the additional structure on the differential forms (the differential operator and the grading operator) come from in this more geometric, supermanifold picture. To see this relationship, we're going to first look at a few examples of the S-points of some particular generalized supermanifolds.

2.4 Calculating S-points and comparing supermanifolds

Proposition 2. $\underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \mathbb{R}^{1|1}$.

Proof. We will prove this by comparing S-points. We want to show there is a bijection

$$\underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) \longleftrightarrow \mathbb{R}^{1|1}(S)$$

which is natural in $S \in SMan$.
On the left hand side, we have

$$\begin{aligned} \underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) &= SMan(S, \underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})) \\ &= SMan(S \times \mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \\ &\cong SAlg(C^\infty(\mathbb{R}^{0|1}), C^\infty(S \times \mathbb{R}^{0|1})) \\ &= SAlg(\Lambda[\theta], C^\infty(S)[\theta]) \end{aligned}$$

The isomorphism between lines 2 and 3 works by sending $g \in SMan(S \times \mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ to

$$\begin{aligned} g^* &: \Lambda[\theta] \to C^\infty(S)[\theta] \\ \theta &\mapsto g^*(\theta) = g_1 + g_0\theta \end{aligned}$$

where $g_1 \in C^\infty(S)^{odd}, g_0 \in C^\infty(S)^{ev}$.
On the right hand side, we have

$$\begin{aligned} \mathbb{R}^{1|1}(S) &= SMan(S, \mathbb{R}^{1|1}) \\ &\cong SAlg(C^\infty(\mathbb{R}^{1|1}), C^\infty(S)) \\ &= C^\infty(S)^{odd} \times C^\infty(S)^{ev} \end{aligned}$$

The desired bijection is given by taking

$$\begin{aligned} \underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(S) &\longleftrightarrow \mathbb{R}^{1|1}(S) \\ g &\longmapsto (g_1, g_0) \end{aligned}$$

□

Actually, $\underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ has more structure: composition gives it a monoidal structure. If we pay attention to the monoidal structure, we get:

Proposition 3. $\underline{SM}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}$ as monoids. The monoidal structure works as: $g \circ g' \mapsto (g_1 + g_0 g_1', g_0 g_0')$.

Recall that every supermanifold is isomorphic to ΠE for some vector bundle E. The S-points of a vector bundle are given in the following proposition:

Proposition 4. Let E be a vector bundle of rank q over a manifold X of dimension p. Then the S-points of the $p|q$-supermanifold ΠE are

$$\Pi E(S) := \{(x, v) | x \in X(S), v \in E_x^{odd}\}$$

where $E_x := C^\infty(S, x^*E)$.

(See Stephan's notes on *Equivariant de Rham cohomology and gauged field theories*, Example 5.34.2, for more details.[4])

In particular, for the tangent bundle $TX \to X$ we have $(\Pi TX)(S) = \{(x,v) | x \in X(S), v \in T_x X^{odd}\}$.

Proposition 5. For any $X \in SMan, \underline{SMan}(\mathbb{R}^{0|1}, X) \cong \Pi TX$.

Proof. We again show this using the S-point formalism. Please see Propn 5.24 in Stephan's notes for details. [4] □

Note that $\underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ acts on $\underline{SMan}(\mathbb{R}^{0|1}, X)$ by precomposition:

Proposition 6. Let $X \in SM$.

$$\mu : \underline{SMan}(\mathbb{R}^{0|1}, X) \times \underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1}) \to \underline{SMan}(\mathbb{R}^{0|1}, X)$$

is given on S-points by

$$((x,v),(g_1,g_0)) \mapsto (x + g_1 v, g_0 v)$$

2.5 Differential forms

Recall that $\Pi TX = (X, C^\infty(\Pi TX)) = (X, \Omega^*(X))$. Also, we saw in Propn 5 that $\Pi TX \cong \underline{SMan}(\mathbb{R}^{0|1}, X)$.

$\Rightarrow$ So $C^\infty(\underline{SMan}(\mathbb{R}^{0|1}, X)) \cong \Omega^*(X)$, as superalgebras.

The extra structure that $\Omega^*(X)$ has (the differential operator, and the $\mathbb{Z}$-grading) correspond to the generators of $Lie(\underline{Diff}(\mathbb{R}^{0|1}))$.

Note that $\underline{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times \subset \underline{SMan}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ is a Lie group. In what follows, let $G = \underline{Diff}(\mathbb{R}^{0|1}), M = \underline{SM}(\mathbb{R}^{0|1}, X)$.

Proposition 6 gave us an action of G on M: $\mu : M \times G \to M$. From this, we can get an action of G on $C^\infty(M) : C^\infty(M) \times G \to C^\infty(M)$, by sending $(f,g) \mapsto (m \mapsto \mu(m,g) \stackrel{f}{\mapsto} \mathbb{R})$.

This gives a Lie group homomorphism: $G \to C^\infty(M)$. Differentiating this gives a Lie algebra homomorphism: $Lie(G) \to End(C^\infty(M))$.

Proposition 7. $Lie(G) = \mathbb{R}\langle N, Q\rangle$, where N is the even generator, Q is the odd generator.

(For more details, again see Stephan's notes, Propositions 5.35, 5.36. [4])
The action of $Lie(G) = \mathbb{R}\langle N, Q\rangle$ on $C^\infty(M) = \Omega^*(X)$ gives the desired operators:

Proposition 8.

$$\begin{aligned} Lie(G) &\longrightarrow End(\Omega^*(X)) \\ N &\longmapsto (\omega \mapsto k\omega) \\ Q &\longmapsto (\omega \mapsto d\omega) \end{aligned}$$

for $\omega \in \Omega^k(X)$.

References

[1] H. Hohnhold, S. Stolz and P. Teichner. *From minimal geodesics to supersymmetric field theories*, A celebration of the mathematical legacy of Raoul Bott, 207274, CRM Proc. Lecture Notes, 50, Amer. Math. Soc., Providence, RI, 2010.

[2] H. Hohnhold, M. Kreck, S. Stolz and P. Teichner. *Differential forms and 0-dimensional supersymmetric field theories*, Quantum Topol. 2 (2011), no. 1, 141

[3] S. Stolz, P. Teichner. *Supersymmetric field theories and generalized cohomology.*, 2011.

[4] S. Stolz, *Equivariant de Rham cohomology and gauged field theories*, 2013.

2.6 Problem Session:

Exercise 1. Work through/discuss the details of Propns 5, 7, 8.

Exercise 2. (For Hari:) Let A be a superalgebra. Define a bracket on A by $[-,-] : Der(A) \otimes Der(A) \to Der(A), [D, E] := D \circ E - (-1)^| D||E|E \circ D$. Show that this gives $Der(A)$ the structure of a super Lie algebra. [From [4], Hmwk 2.8.]

Definition 5. A super Lie algebra is a $\mathbb{Z}/2$-graded vector space $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is

1. skew-symmetric in the graded sense: $[a, b] = -(-1)^{|a||b|}[b, a]$, for homogeneous elements $a, b \in \mathfrak{g}$
2. satisfies the graded version of the Jacobi identity: $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$

Definition 6. Let A be a superalgebra. A linear map $D : A \to A$ (not necessarily parity preserving) is a derivation of parity $|D| \in \mathbb{Z}/2$ if

$$D(ab) = (Da)b + (-1)^{|D||a|}a(Db).$$

Write $Der(A) = Der^{ev}(A) \oplus Der^{odd}(A)$, a $\mathbb{Z}/2$-graded vector space.