Notre Dame Graduate Student Topology Seminar, Spring 2018

Lecture 3: Field theories over a manifold, and smooth field theories

As mentioned last time, the goal is to show that

$$0|1\text{-TFT}^n(X) \cong \Omega^n_{cl}(X),$$

$$0|1\text{-TFT}^n[X] \cong H^n_{dR}(X)$$

Last time, we showed that:

$$\begin{aligned}\Omega^*(X) &\cong C^\infty(\Pi TX)\\ &\cong C^\infty(\underline{SMan}(\mathbb{R}^{0|1}, X)\end{aligned}$$

Today I want to introduce two other ingredients we'll need:

- field theories over a manifold
- smoothness of field theories via fibered categories

3.1 Field theories over a manifold

Field theories over a manifold X were introduced by Segal. The idea is to give a family of field theories parametrized by X. Note that this is a general mathematical move that is familiar; for example, instead of just considering vector spaces, we consider families of vector spaces, i.e. vector bundles, which are families of vector spaces parametrized by a space, satisfying certain additional conditions. In the vector bundle example, we can recover a vector space from a vector bundle E over a space X by taking the fiber E_x over a point $x \in X$. The first thing we will discuss is how to extract a field theory as the "fiber over a $x \in X$" from a family of field theories parametrized by some manifold X.

Recall that in Lecture 1, we discussed the non-linear sigma-model as giving a path integral motivation for the axioms of functorial field theories. The 1-dimensional non-linear sigma model with target M^n did the following:

$$\begin{aligned}\sigma_M : 1 - RBord &\longrightarrow Vect\\ pt &\longmapsto C^\infty(M)\\ [0,t] &\longmapsto e^{-t\Delta_M} : C^\infty(M) \to C^\infty(M)\end{aligned}$$

The Feynman-Kan formula gave us a path integral description of this operator: for $x \in M$, we had

$$(e^{-t\Delta} f)(x) = \int_{\{\phi:[0,t]\to X | \phi(t)=x\}} f(\phi(0)) \frac{e^{S(\phi)}\mathcal{D}\phi}{Z}$$

If instead of having just one target manifold M, how does this change when one has a whole family of manifolds $\{M_x\}$?

A first naive approach might be to simply take this X-family of Riemannian manifolds, $\{M_x\}$. However, this does not depend smoothly on X so we wouldn't be taking advantage of all of the data that we can introduce here. Instead (analogous to the move from vector spaces to vector bundles), we want to produce a fiber bundle out of these manifolds, i.e. we want to set $E := \bigsqcup_{x\in X} M_x \xrightarrow{\pi} X$.

Note that this total space has an induced Riemannian metric. From this additional structure of the fiber bundle, we get additional structure when we translate to the functorial field theory picture: we want:

$$\begin{gathered} E : 1\text{-RBord}(X) \longrightarrow Vect \\ objects : x \mapsto C^\infty(M_x) \\ morphisms : ([0,1] \xrightarrow{\gamma} X) \mapsto E(\gamma) : C^\infty(M_{\gamma(0)}) \to C^\infty(M_{\gamma(t)}) \end{gathered}$$

To define this $E(\gamma)$, we want to generalize the Feynman-Kac formula: for $f \in C^\infty(M_x), z \in M_y$,

$$((E(\gamma))f)(z) = \int_{\{\phi:[0,t]\to M|\pi\circ\phi=\gamma\}} f(\phi(0))\frac{e^{S(\phi)}\mathcal{D}\phi}{Z}$$

This type of generalization of the non-linear sigma model inspires the definition of a *field theory over X*.

Definition 1. Let X be a manifold. A *field theory over X* is a symmetric monoidal functor

$$E : d\text{-Bord}(X) \to Vect$$

where d-Bord(X) is the bordism category as before, where all objects and morphisms are are equipped with smooth maps to X. (Note that having a bordism $\Sigma \xrightarrow{f} X$ from $Y_0 \xrightarrow{f_0} X$ to $Y_1 \xrightarrow{f_1} X$ then means that the f agrees with f_0, f_1 when restricted to the corresponding boundaries.)

Example 1. To see what this idea of a field theory over X gives us, consider 0-dimensional field theories:

$$0\text{-TFT}(X) = Fun(0\text{-Bord}(X), Vect).$$

Note that there is only one (-1)-dimensional object: $\emptyset$. This is the monoidal unit in the bordism category, so it must be sent to $\mathbb{R}$: $E(\emptyset) = \mathbb{R}$.
Next, consider the morphisms: 0-Bord$(X)(\emptyset,\emptyset) \mapsto \mathbb{R}$. Remember that 0-Bord$(\emptyset,\emptyset)$ are diffeomorphism classes of 0-dimensional bordisms, so here we're just looking at disjoint unions of points. Becuase a TFT is a symmetric monoidal functor, its action will be completely determined by what it does on one point. So on the left hand side, the relevant bordism to look at is $\{pt \to X\} \cong X$. Because of monoidal unit reasons, we know E must send this to $\mathbb{R}$. Thus, 0-TFT$(X) \cong Maps(X, \mathbb{R})$. Note that there is no smoothness or continuity requirement on these maps!

We would like to be able to have smoothness or continuity requirements on the maps we saw above...how would we incorporate this? What would it mean for E to be smooth? Heuristically, we want $E(Y), E(\Sigma)$ to depend smoothly on Y, Σ, respectively. To implement this, we will replace the the previous source and target categories of our field theories d-Bord, $Vect$ by their family versions.

3.2 Family versions of categories

In what follows, we will use the language of fibered categories as in Section 2.7 of Stolz-Teichner's paper "Supersymmetric field theories and generalized cohomology." We first recall the definition:

Definition 2. Let B and S be categories. We say that a functor $p : B \to S$ is a *(Grothendieck) fibration* if pull-backs exist. That is, for every object $Z \in B$ and arrow $f : S \to T = p(Z)$ in S, there is a cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Z \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & T. \end{array}$$

A *fibered category over* S is a category B together with a fibration $B \to S$.

The following example illustrates how one can extract the fiber over an object using fibered categories.

Example 2. Let $s \in S$ be an object and let $p : B \to S$ be a category fibered over S. . We define a subcategory B_s of B by setting the objects to be the set of $b \in B$ such that $p(x) = s$, and the morphisms to be the set of maps $f : b \to b'$ such that $p \circ f = id_s$.

Another example is described in Section 2.7 of Stolz-Teichner:

Example 3. The forgetful functor $p : Bun \to Man$ sending a smooth vector bundle $Y \to S$ to the underlying manifold S is a Grothendieck fibration. The key point is that one can construct the desired pullbacks using the pullback of fiber bundles.

We can use fibered categories to make sense of field theories parametrized by a manifold X.

Definition 3. Define the category d-Bord^f parametrized over Man as follows. The objects are families of d-manifolds over a manifold, i.e. fiber bundles $p : Y \to S$ where the fibers are closed (oriented) (d-1)-manifolds. The morphisms are families of bordisms $\Sigma \to S$.

Definition 4. Define the category $Vect$ parametrized over Man as follows. The objects are families of vector spaces over a manifold S, i.e. a vector bundle $V \to S$. The morphisms are families of linear maps.

In this language, a TFT is then a fibered (symmetric monoidal) functor between these fibered categories.

In order to extend this definition to $d|\delta$-TFT's, we want to use the language of stacks. Before doing so, we need to discuss Grothendieck topologies and sites.

Definition 5. Let C be a category. A *Grothendieck topology on* C is an assignment to each object $U \in C$ a collection of sets of arrows $\{U_i \to U\}$, called *coverings of* U, such that the conditions in Definition 2.24 of Vistoli's "Notes on Grothendieck topologies, fibered categories, and descent theory" are satisfied. Roughly speaking, these conditions say that open coverings are compatible with each other in the way classical open covers of manifolds are compatible.

A category equipped with a Grothendieck topology is a *site*.

We can now define stacks.

Definition 6. Let $B \to S$ be a fibered category on a site S. We say that B is a *stack over* S if for each covering $\{U_i \to U\}$ in S, the functor $B(U) \to B(\{U_i \to U\})$ is an equivalence of categories.

Here, the notation $B(-)$ should be thought of in the case where $B = Bun$ and $S = Man$. In this case, $B(U)$ associates to a manifold U the fiber bundles over it and the condition that $B(U) \to B(\{U_i \to U\})$ be an equivalence of categories says that a fiber bundle over U can be built out of fiber bundles on some open cover $\{U_i \to U\}$. Equivalently, one can also think of a stack as a pseudofunctor from a site S to the 2-category of categories which satisfies descent for all covers.

In particular, we can recover a fibered category from a stack by forgetting about the Grothendieck topology data on the site. We will occasionally define fibered categories as psuedofunctors, i.e. by specifying categories C_0 and C_1 fibered over S with functors between them.

Now, we'd like to define families of TFT's parametrized by a generalized supermanifold X. The naive extension would be to say that the source category for these TFT's is the category of $d|\delta$-dimensional bordisms with maps to X. However, this runs into the same problem as above since it doesn't require smoothness anywhere. For today, let $SMan$ denote the category of generalized supermanifolds.

Definition 7. We define the fibered category of $d|\delta\text{-Bord}^f \to SMan$ as a pseudofunctor as follows. Let $S \in SMan$. Define $(d|\delta\text{-Bord}^f)_0(S)$ to be the category with objects fiber bundles $p : Y \to S$ with fibers $(d-1)|\delta$-manifolds, and morphisms the obvious commuting squares. Define $(d|\delta\text{-Bord}^f)_1(S)$ to be the category with objects fiber bundles $\Sigma \to S$ with fibers $d|\delta$-bordisms, and morphisms commuting squares.

Example 4. When $d = 0$ and $\delta = 1$, we have

$$(0|1 - Bord^f)_0(S) = \{\emptyset \to S\},$$

$$(0|1 - Bord^f)_1(S) = \{\Sigma \to S : \text{ fibers are } 0|1\text{-closed bordisms}\}.$$

Definition 8. Let $X \in \text{Man}$. A *$d|\delta$-dimensional (smooth) supersymmetric field theory over* X is a fibered symmetric monoidal functor over the category of generalized supermanifolds:

$$E : d|\delta\text{-Bord}^f(X) \to Vect^f.$$

Example 5. Let's see what this extra structure gives us.

1. Let $E : d|\delta\text{-Bord} \to Vect$ be a field theory. Take Σ a closed d-manifold. Then we know that $E(\Sigma) \in \mathbb{R}$.

2. Now let's add the structure of being a field theory over X: Fix $X \in Man$. To Σ, we add a map $f : \Sigma \to X$. Now $E : d|\delta\text{-Bord}(X) \to Vect$, will map $E(\Sigma, f) \in \mathbb{R}$.

3. Now let's add the structure of being a *supersymmetric (smooth)* field theory over X: Let $E : d|\delta\text{-Bord}^f(X) \to Vect^f$. For this one, let's change our source bordism a bit; now consider

$$\begin{array}{ccc} map(\Sigma, X) \times \Sigma & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle \pi} & & \\ map(\Sigma, X) & & \end{array}$$

Here π exhibits $map(\Sigma, X) \times \Sigma$ as a $map(\Sigma, X)$-family of supermanifolds; in addition, they are equipped with a smooth map to X.

Applying E to this, we get

$$E\left(\begin{array}{ccc} map(\Sigma, X) \times \Sigma & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle \pi} & & \\ map(\Sigma, X) & & \end{array} \right) \in C^\infty(map(\Sigma, X), \mathbb{R}).$$

Example 6. Taking $\Sigma = \mathbb{R}^{0|1}$ in the last example, we get $E(-) \in C^\infty(\Pi TX, \mathbb{R}) = \Omega^*(X)$.

Note that (along similar lines to the argument in example 1), in the case of $0|1\text{-TFT}(X)$, the only $-1|1$-dimensional object is $\emptyset$; and so the $0|1$-dimensional bordisms are all disjoint unions of points; and because of symmetric monoidal reasons, it suffices to look at what the field theory does to one point, $\mathbb{R}^{0|1}$. In other words, $E \in 0|1\text{-TFT}(X)$ is determined by what it does on

$$\begin{array}{ccc} map(\mathbb{R}^{0|1}, X) \times \mathbb{R}^{0|1} & \xrightarrow{ev} & X \\ \downarrow{\scriptstyle \pi} & & \\ map(\mathbb{R}^{0|1}, X). & & \end{array}$$

We will identify this action through a series of steps.

First, note that if $M \in SMan$ and G is a super Lie group with $G \curvearrowright X$, then we can form the quotient stack $M//G \to SMan$ which is in particular a fibered category of $SMan$. For this, we will need our identification that we established last time: $\underline{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times$. Then we have

$$(M//G)_0(S) = \{\text{principal } G\text{-bundles over } S, \text{ with a } G\text{-equivariant map from the total space to } M\},$$

$$(M//G)_1(S) = \{\text{maps of principal } G\text{-bundles}\}.$$

Last time, we saw that for X a supermanifold, there is an action

$$G = \underline{Diff}(\mathbb{R}^{0|1}) \curvearrowright M = \underline{SMan}(\mathbb{R}^{0|1}, X).$$

We claim that the quotient stack $M//G$ is a substack of $0|1 - Bord^f(X)$. Indeed, it is not hard to see that there is a bijection between principal G-bundles and diagrams of the form

$$\begin{array}{ccc} P \times_G \mathbb{R}^{0|1} & \longrightarrow & X \\ \downarrow & & \\ S. & & \end{array}$$

Now, it is a fact (which we will not prove) that $M//G$ freely generates $0|1 - Bord^f(X)$ as a symmetric monoidal stack. One can use this to prove the following:

Proposition 1 (Propn 5.5 in HKST)**.** Let $X \in Man$.

$$\begin{aligned} 0|1\text{-TFT}(X) &= Fun^{\otimes}(0|1\text{-Bord}^f(X), \underline{R}) \\ &\cong Fun_{SMan}(\underline{SMan}(\mathbb{R}^{0|1}, X)//\underline{Diff}(\mathbb{R}^{0|1}), \underline{\mathbb{R}}) \\ &\cong SMan(\underline{SMan}(\mathbb{R}^{0|1}, X), \underline{\mathbb{R}})^{\underline{Diff}(\mathbb{R}^{0|1})} \\ &\cong \Omega^0_{cl}(X) = \{f \in C^\infty(X) | df = 0\} \end{aligned}$$

Proof. Recall identification we made last week

$$\underline{Diff}(\mathbb{R}^{0|1}) \cong \mathbb{R}^{0|1} \rtimes \mathbb{R}^\times.$$

We showed in the exercise session that

$$Lie(\underline{Diff}(\mathbb{R}^{0|1})) \cong \mathbb{R}\langle t\frac{d}{dt}, t\frac{d}{d\theta}\rangle.$$

In particular, given $\omega \in \Omega^k(X)$, the first operator sends it to $k \cdot \omega$ and the second operator sends it to $d\omega$.

Claim: the invariants of the $\underline{Diff}(\mathbb{R}^{0|1})$ action are the same as the things annihilated by the $Lie(\underline{Diff}(\mathbb{R}^{0|1}))$ action.

Since the kernel of the action of $Lie(\underline{Diff}(\mathbb{R}^{0|1}))$ consists of closed 0-forms, we conclude the proposition. □

3.3 Exercises

1. Work out the details of how the fibered bordism category is a pseudofunctor. How do psuedofunctors relate to stacks and fibered categories? In the case of the $0|1$-dimensional bordism category, write these details out more explicitly (what is the category we're looking at in this case?).

2. Show that

$$Fun(\underline{SMan}(\mathbb{R}^{0|1}, X)//\underline{Diff}(\mathbb{R}^{0|1}), \underline{\mathbb{R}}) \cong SMan(\underline{SMan}(\mathbb{R}^{0|1}, X), \mathbb{R})^{\underline{Diff}(\mathbb{R}^{0|1})}.$$

3. Show that

$$SMan(\underline{SMan}(\mathbb{R}^{0|1}, X), \mathbb{R})^{\underline{Diff}(\mathbb{R}^{0|1})} \cong \Omega^*(X)^{\underline{Diff}(\mathbb{R}^{0|1})}.$$

   In particular, show that it suffices to look at the annihilator of the Lie algebra action.