Notre Dame Graduate Student Topology Seminar, Spring 2018

# Lecture 4: Twisted field theories

Recall: Last time we showed that

$$0|1\text{-TFT}(X) \cong \Omega^0_{cl}(X)$$

The goal for today is to introduce *twisted* field theories. These are particularly relevant for physical application of functorial field theories; also, they will allow us to get non-zero degree differential forms in the above classification result.

## 4.1 Motivation

Recall that our motivation for the definition of functorial field theories came from the path integral point of view: we are viewing field theories as functors from $Bord \to Vect$, which take a closed manifold $\Sigma$ to a number: this number should be given by the path integral over all of the fields on $\Sigma$.

The simplest situation is that of a free field theory (also called a linear sigma-model), where the target space $V$ for the fields is an inner product space. This allows us to think of the space of fields as a vector space $\mathcal{F} = Maps(\Sigma, V)$. For free field theories, the action functional is given by a quadratic functional on the space of fields: for $\phi \in Maps(\Sigma, V)$, the action function sends that to $S(\phi) = ||\phi||^2$.

If we simplify the situation still further, and consider the case where the space of fields is finite dimensional, $\mathcal{F} = \mathbb{R}^n$, then the quadratic functional giving the action functional can be associated to a matrix. In this case:

$$\int_{\mathcal{F}} e^{-\langle \phi, A\phi \rangle} \mathcal{D}\phi = (det(A))^{-1/2}$$

But then the problem arise of how to make sense of this when we pass to the real world, where the space of fields is infinite dimensional. How can one make sense of the determinant in this case?

Quillen showed the following: as long as $A$ is an elliptic operator (i.e. linear differential operator with an invertible principal symbol), one can always make sense of the determinant. However, the determinant of $A$ no longer gives a number, but rather an element of a line (the line is dependent on $A$).

Upshot: We want field theories (inspired by the path integral approach) to behave similarly. To a closed manifold $\Sigma$, our field theory $E$ should associate a generalized determinant– i.e. not a number, but an element of a line associated to $\Sigma$. So what we're moving towards are sections of a line bundle...

## 4.2 Definition of twisted field theory

To do this, we will need to incorporate more categorical levels into our definition of field theories. First, a definition:

**Definition 1.** A *twist* is a (symmetric monoidal, fibered) functor

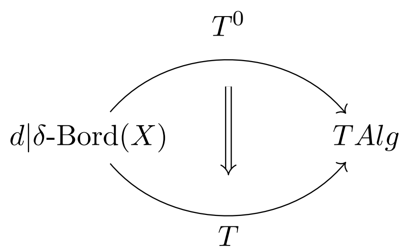
$$T : d|\delta\text{-Bord}(X) \to TAlg.$$

Note: the target category here is topological algebras (objects are topological algebras, and the morphisms are bimodules). We need to use this category because we're moving up the categorical ladder (from a field theory giving a number, to giving an element of a vector space).

**Example 1.** The *trivial twist*, $T^0$, is the constant functor which sends every $Y \in d|\delta\text{-Bord}(X)_0$ to $T^0(Y) = \mathbb{R}$, and every $\Sigma \in d|\delta\text{-Bord}(X)_1$ to $T^0(\Sigma) = \mathbb{R}$ (as an $\mathbb{R} - \mathbb{R}$-bimodule).

Given a twist $T$ as above, we can now define a twisted field theory:

**Definition 2.** A $T$-twisted $d|\delta$-TFT over $X$ is a natural transformation $E$:

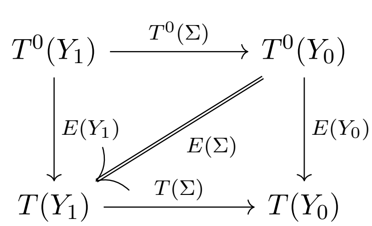



Let's unpack this definition a bit. For now, for simplicity, let $X = pt$ and ignore the fibered/family aspect of the categories by just looking at the fibered categories over the point.

The twist $T$ is a functor: it associates to an object $Y$ a topological algebra $T(Y)$; to a bordism $\Sigma$ from $Y_0$ to $Y_1$, it associates a $T(Y_0) - T(Y_1)$-bimodule $T(\Sigma)$.

What does the natural transformation $E$ do? To every object $Y \in d|\delta\text{-Bord}(X)$, $E$ gives a morphism in $TAlg$: $E(Y) : T^0(Y) \to T(Y)$. (Note: one can also think of this morphism as a bimodule $E(Y)$: this is a left module over $T(Y)$ and a right module over $T^0(Y) = \mathbb{R}$; i.e. $E_0(Y)$ is a left $T(Y)$-module.)

To every morphism (i.e. bordism) $\Sigma : Y_0 \to Y_1$, $E$ associates a natural transformation (note that here we're taking advantage of the fact that we're working with 2-categories; we don't want to require the commutativity of the following square, but rather just the existence of a 2-morphism for it):



Note: the twists $T^0, T$ act contravariantly on $\Sigma$; we define it this way to get what we want in the end (see the 'sanity check' below).

This $E(\Sigma)$ is a map between $T(Y_0) - T^0(Y_1)$-bimodules; i.e. a map of left $T(Y_0)$-modules. Another way to write $E(\Sigma)$ is:

$$E(\Sigma) : {}_{T(Y_0)}E(Y_0) \longrightarrow {}_{T(Y_0)}T(\Sigma) \otimes_{T(Y_1)} E(Y_1)$$

Let's do a sanity check:

(i) If we take $T = T^0$, we would want to get our old definition of a field theory. We get $E(\Sigma) : E(Y_0) \to E(Y_1)$, a linear map; this is indeed what we had before in our old definition of a (non-twisted) field theory.

(ii) If $\Sigma$ is closed, then $Y_0 = \emptyset = Y_1$. Let's check that we get something that resembles the generalized determinant that we wanted. Here $E(\Sigma) : \mathbb{R} \to T(\Sigma)$, giving us an element of the vector space $T(\Sigma)$; this is what we wanted from our generalization of the determinant.

### 4.3 Applying to our specific example

Now let's move towards look at how this applies to our specific example of $0|1$-TFT$(X)$. In this case, what is the source category of the twist functors?

$0|1$-Bord$(X)$ has trivial objects (the only $-1|1$-dimensional manifold is $\emptyset$): the objects are fiber bundles: $\begin{array}{ccc} S \times \emptyset & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$.

The morphisms are fiber bundles $\begin{array}{ccc} \Sigma & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$ where $\Sigma$ is a fiber bundle with $0|1$-dimensional closed fibers.

Note that because we're working with symmetric monoidal categories, it suffices to consider fiber bundles with fiber $\mathbb{R}^{0|1}$; all other $0|1$-dimensional fibers will be disjoint unions of this one.

Also, note that we don't lose information by considering the globally trivial bundle:

$$\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{f} & X \\ \downarrow & & \\ S & & \end{array}$$

Note: There is a universal $S$ we can use. Notice that the only piece of information that we're given in the above bundle is the map $f : S \times \mathbb{R}^{0|1} \to X$, i.e. $f \in \underline{SMan}(\mathbb{R}^{0|1}, X)(S)$. Take $f$ to be the identity map; i.e. take $S = \underline{SMan}(\mathbb{R}^{0|1}, X)$.

**Claim:** This is the "universal $S$-family" for bundles of this type. In other words, any bundle of the form $\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{f} & X \\ \downarrow & & \\ S & & \end{array}$ is obtained as a pullback of $\begin{array}{ccc} \underline{SMan}(\mathbb{R}^{0|1}, X) \times \mathbb{R}^{0|1} & \xrightarrow{ev} & X \\ \downarrow & & \\ \underline{SMan}(\mathbb{R}^{0|1}, X) & & \end{array}$ $=\star$

Upshot: We don't need to consider all $S$-families; it suffices to consider this universal $S$-family (similar to how things work with classifying spaces). If we know how the twist $T$ acts on this, then we know how $T$ acts on all objects (and similarly for a $T$-twisted field theory $E$).

Recall what $T$ should give us (remember, we're dealing with the family/fibered category version here): $T(\star)$ should be an equivariant vector bundle (with respect to the $\underline{Diff}(\mathbb{R}^{0|1})$ action) over $\underline{SMan}(\mathbb{R}^{0|1}, X)$: $\mathcal{V} \to \underline{SMan}(\mathbb{R}^{0|1}, X)$.

What should $E(\star)$ be? If we ignore the family version aspect for a minute, remember that for a closed bordism $\Sigma$, we get $E(\Sigma) : E(Y_0) = E(\emptyset) = \mathbb{R} \to T(\Sigma) = T(\Sigma) \otimes_{T(Y_1)} E(Y_1)$; in other words, $E(\Sigma) \in T(\Sigma)$.

When we incorporate the family version of things, this then becomes: $E(\star) \in \Gamma(T(\star))^{Diff(\mathbb{R}^{0|1})}$, the $Diff(\mathbb{R}^{0|1})$-equivariant sections).

So now let's define what this vector bundle $T(\Sigma)$ is.

First, note some general facts: If we have an action $G \times M \to M$ and a representation of $G$ on a finite dimensional vector space $V$, $\phi : G \to Aut(V)$, then we can form a $G$-equivariant vector bundle: $\begin{array}{c} M \times V \\ \downarrow \\ M \end{array}$ (The $G$-action on the total space is given by the diagonal action($(g, m, v) \mapsto (gm, gv)$).) Note that this is an equivariant bundle; but it is not equivariantly trivial.

The equivariant sections of this bundle are:

$$\begin{aligned} \Gamma(M, M \times V)^G &\cong \Gamma(M, V)^G \\ &\cong (C^\infty(M) \otimes V)^G \end{aligned}$$

In our case, take $G = Diff(\mathbb{R}^{0|1}), M = \underline{SMan}(\mathbb{R}^{0|1}, X)$. The action of $G$ on $M$ was defined in lecture 3. It remains to choose a representation. Choose the representation:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & Aut(\mathbb{R}) \\ & {\scriptstyle pr_2} \searrow \quad \nearrow {\scriptstyle m} & \\ & \mathbb{R}^\times & \end{array}$$

The arrow $m$ comes from a multiplication action on $\mathbb{R}$. Explicitly, it is:

$$\begin{aligned} \mathbb{R}^\times \times \mathbb{R}_1 &\longrightarrow \mathbb{R}_1 \\ (\alpha, t) &\longmapsto \alpha^{-1} t \end{aligned}$$

Then consider the $Diff(\mathbb{R}^{0|1}$-equivariant bundle:

$$\begin{array}{c} \mathcal{L} := M \times \mathbb{R}_1 \\ \downarrow \\ M \end{array}$$

Define $T(\star)$ to be this bundle.

Notice that the symmetric monoidal structure of $TAlg$ allows us to take tensor products of the twist functors. For this twist functor, $T^{\otimes n}(\star)= \begin{array}{c} \mathcal{L}^{\otimes \mathrm{n}} := M \times \mathbb{R}_n \\ \downarrow \\ M \end{array}$.

We use $\mathbb{R}_n$ to denote the particular action of $\mathbb{R}^\times$:

$$\begin{aligned} \mathbb{R}^\times \times \mathbb{R}_n &\longrightarrow \mathbb{R}_n \\ (\alpha, t) &\longmapsto \alpha^{-n} t \end{aligned}$$

Write $E^n$ for the $T^{\otimes n}$-twisted field theory we want to define. We know that $E^n \in \Gamma(M, M \times \mathbb{R}_n)^G$.

We get the following:

$$\begin{aligned} \Gamma(M, \mathcal{L}^{\otimes n})^G &= \Gamma(M, M \times \mathbb{R}_n)^G \\ &= (C^\infty(M) \otimes \mathbb{R}_n)^G \\ &= (\Omega^*(X) \otimes \mathbb{R}_n)^{\mathbb{R}^{0|1} \rtimes \mathbb{R}^\times} \\ &= ((\Omega^*(X) \otimes \mathbb{R}_n)^{\mathbb{R}^{0|1}})^{\mathbb{R}^\times} \\ &= (\Omega^*_{cl}(X) \otimes \mathbb{R}_n)^{\mathbb{R}^\times} \end{aligned}$$

(The last line follows because action of $\mathbb{R}^{0|1}$ on $\mathbb{R}_n$ is trivial, so we can ignore that one and just look at action on $\Omega^*(X)$.)

What is the $\mathbb{R}^\times$ action on $\Omega^*_{cl}(X) \otimes \mathbb{R}_k$? Take $\omega \in \Omega^k_{cl}(X), t \in \mathbb{R}_n, \alpha \in \mathbb{R}^\times$. Remember that $\mathbb{R}^\times$ acted on $\Omega^*(X)$ as the grading operator. So we get:

$$\begin{aligned} \alpha(\omega \otimes t) &= \alpha^k \omega \otimes \alpha^{-n} t \\ &= \alpha^{k-n}(\omega \otimes t) \end{aligned}$$

If we want the invariant part of this, we need to take $k - n = 0 \Rightarrow n = k$. So $\omega \in \Omega^n_{cl}(X)$.

This completes our classification:

**Proposition 1.** For $X \in Man$,

$$0|1\text{-TFT}^n(X) \cong \Omega^n_{cl}(X)$$

### 4.4 Problems

**Exercise 1.** Double check the order that we want for the bimodules: for $\Sigma : Y_0 \to Y_1$, do we get ${}_{T(Y_0)}T(\Sigma)_{T(Y_1)}$, or is the bimodule the other way around?

**Exercise 2.** Prove the claim that $\star$ is the "universal $S$-family" of the fiber bundles we're looking at. I.e., any bundle of the form $\begin{array}{ccc} S \times \mathbb{R}^{0|1} & \xrightarrow{f} & X \\ \downarrow & & \\ S & & \end{array}$ is obtained as a pullback of

$$\begin{array}{ccc} \underline{SMan}(\mathbb{R}^{0|1}, X) \times \mathbb{R}^{0|1} & \xrightarrow{ev} & X \\ \downarrow & & \\ \underline{SMan}(\mathbb{R}^{0|1}, X) & & \end{array} = \star$$

**Exercise 3.** Work out the details of the following claim: the equivariant sections of the bundle $M \times V \to M$ are (show why the second line holds):

$$\begin{aligned} \Gamma(M, M \times V)^G &\cong \Gamma(M, V)^G \\ &\cong (C^\infty(M) \otimes V)^G \end{aligned}$$