Talk on TFT's

Goal: "1-TFT's are ~~x~~ related to KO-Theory"

Recall: $KO^0(X) = \cancel{R}\ \mathrm{Gr}\left(\left\{\begin{matrix}\text{vect bundles} \\ \text{of } X\end{matrix}\right\}/\text{iso}, \oplus\right) = [X_+, BO\times\mathbb{Z}]$

as $X \to BO$ is a stable iso class of a vect bundle, and

$\mathbb{Z}$ remembers the dimension.

~~Goal: "1-TFT~~

Motivational example: 1-TFT's.

$1\text{-Bord} \xrightarrow{E} \text{Vect}$ What can we say about V?

$pt \longmapsto E(pt) = V$

$\rightsquigarrow V$ is self dualizable

$\Rightarrow V$ is finite dim.

A map of 1-TFT's $E \to E' \iff E(pt) \to E'(pt)$ linear map

grp'oid of 1-TFT's $\longleftrightarrow$ grp'oid of fin dim vect spaces. So passing to

spaces: Space of 1-TFT $\simeq \coprod_n BO(n)$

Goal: " space of 1-TFT's w. correct decoration $\simeq BO\times\mathbb{Z}$ "

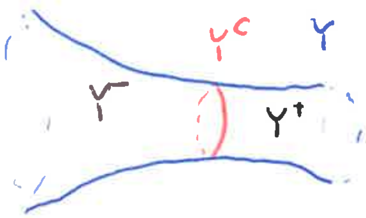
First decoration: Riemannian

What is d-RBord?

obj: $(Y, Y^c, Y^\pm)$ w. Y ~~Eucl~~ Riem d-mfld (usually non-compact),

$Y^c \subset Y$ compact $(d-1)$ submfld, called core of Y.

$Y^\pm = Y^- \sqcup Y^+$ s.t. full dim open subsets of Y w. $Y - Y^c = Y^\pm$



up to the relation $(Y_0, Y_0^c, Y_0^\pm) \sim (Y_1, Y_1^c, Y_1^\pm)$ if $\exists W_i \subseteq Y_i$ open neighborhood of Y_i^c and

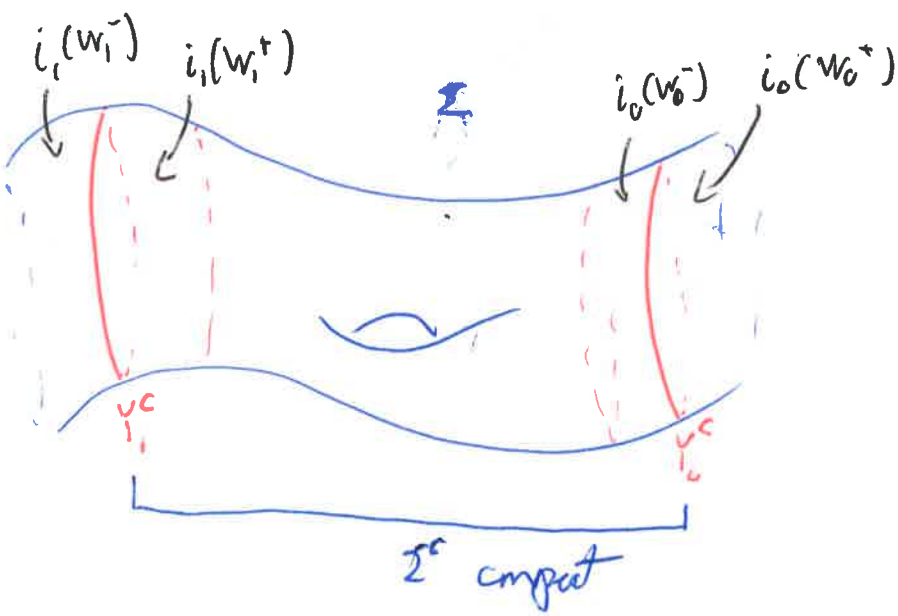
$f: W_0 \to W_1$ invertible isometry w. $f(Y_0^c) \subset Y_1^c$ and $f(W_0^\pm) \subset W_1^\pm$

$W_i^\pm = Y_i^\pm \cap W_i$

So basically the collar around Y^c is not that important

Morph: $(Y_1, Y_1^c, Y_1^\pm) \xleftarrow{(\Sigma, i_0, i_1)} (Y_0, Y_0^c, Y_0^\pm)$. Σ ~~Euclidean~~ Riem d-mfld, smooth maps

$i_j: W_j \to \Sigma$ $\quad W_j \subset Y_j$ open neighborhood of Y_j^c



up to similar relations as above

$1\text{-}\underset{\Sigma}{\mathbb{R}}\text{bord}$ has obj: $(\mathbb{R}, \{0\}, \mathbb{R}_{\pm})$

and bordism ~~$X(\mathbb{R})$~~ $\Sigma = \mathbb{R}$ w. $\mathbb{R}^c = [0,t]$ $t > 0$

$1\text{-}\mathbb{R}\text{bord}^{\sqcup} \longrightarrow \mathrm{Vect}^{\otimes}$

$pt \longmapsto V$ ← Note V is no longer self dual

$pt \xleftarrow{I_t} pt \longmapsto I_t \rightsquigarrow \mathbb{R}_{\geq 0} \xrightarrow{I_{(\cdot)}} \mathrm{End}(V)$ ~~gp~~ monoid maps (it done smth) smooth

What can we say about $I_t : V \to V$?

Def: $\mathcal{C}^{\otimes}$ sym mon, $f \in \mathcal{C}(X, Y)$ is thick if $\exists Z$ and

$X \otimes Z \to \mathbb{1}$ and $\mathbb{1} \to Z \otimes Y$ s.t.

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[d, phantom, "\cong"] & & Y \arrow[d, phantom, "\cong"] \\
X \otimes \mathbb{1} \arrow[r] & X \otimes Z \otimes Y \arrow[r] & \mathbb{1} \otimes Y
\end{tikzcd}$$

Fact: $\mathcal{C} = (\mathrm{Top\,Vect}, \text{w. } \otimes)$ F thick ⟹ F nuclear (stronger than expect)

Note: in 1-RBord I_t thick, so $I_t : V \to V$ ~~thick~~ Nuclear

$\mathbb{R}_{\geq 0} \longrightarrow \overset{\text{Nuclear}}{\mathrm{End}(V)}$ is smooth so $\mathrm{Lie}(\mathbb{R}_{>0}) \longrightarrow \mathrm{End}(V)$

$\frac{d}{dt} \longmapsto A$ ← the essential data of E

Fact: 1-RFT are ~~contractible~~ contractible path components

$1|1$ - ~~R~~ E FT's $\qquad 1|1$ - ~~R~~ E Bord $\longrightarrow$ SVect

$$\mathbb{R}^{0|1} \longmapsto V$$

So in the non-super case we had data of $\mathbb{R}_{>0} \to End(V)$ since $\mathbb{R}_{>0}$ - moduli space of intervals. Now we need a moduli space of super intervals

$\mathbb{R}^{1|1}_+ \to End(V)$ smooth ⟿ $Lie(\mathbb{R}^{1|1}_+) \to End(V)$

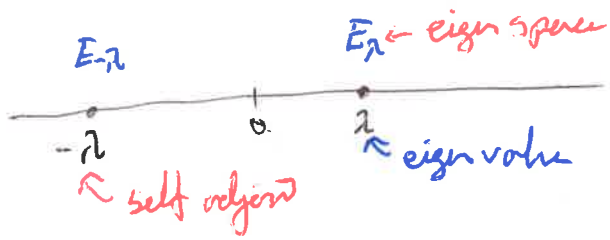
$Lie(\mathbb{R}^{1|1}_+) = \mathbb{R}\langle \underset{\text{even}}{\frac{\partial}{\partial t}}, \underset{\text{odd}}{D} \rangle / [D,D] = -2\frac{\partial}{\partial t}$ = Free Lie gen by odd generator D.

So $Lie(\mathbb{R}^{1|1}_+) \to End(V)$ determined by odd $D \in End(V)$

Thickness $\Rightarrow$ D self adjoint, discrete spectrum w. fin multiplicity

So gp'oid of $1|1$-RFT's = gp'oid of (V,D) as above,

given (V,D) we can construct configuration $Spec\, D \subset \mathbb{R}$



Note; $E_{-\lambda} = \alpha(E_\lambda)$ where α is grading involution.

So gp'oid of $(V,D) \simeq$ space of conf like this

Given sVect V, $sdim\, V = dim\, V^{ev} - dim\, V^{odd}$

$Conf \to \mathbb{Z}$ by $sdim\, E_0$ well def as $sdim(E_{-\lambda} \oplus E_\lambda) = 0$

Surj $\Rightarrow \pi_0\, Conf = \mathbb{Z} \Rightarrow \pi_0\ 1|1$ - ~~R~~ E FT $= \mathbb{Z}$.

Fact: $\mathrm{Conf}^{fin} \subset \mathrm{Conf}$ is a htpy equiv.

So we need to study Conf^{fin}.

Def: Cat Q = obj $\mathbb{C}_2$-graded v.s. $\subset$ $\mathcal{H}$ int dim s Hilbert space

s construction on Vect fin

morph $W_1 \to W_2$ if $W_1 \subset W_2$ and consists of the data

β ~~and~~ on odd ~~automorphism~~ of $W_1^\perp \subset W_2$

Involution $\beta^2 = id$

$BQ \to \mathrm{Conf}^{fin}$ on 0-simplices $W \mapsto \; \longrightarrow\!\!\overset{W}{\underset{0}{\bullet}}\!\!\longrightarrow$

on 1-simplex

$\left\{ \begin{matrix} W_1 \\ \cap \\ W_2 \end{matrix} \;\; \beta \subset W_1^\perp \right\} \longmapsto$ path in Conf^{fin} $[0,\infty] \to \mathrm{Conf}^{fin}$

β splits $W_1^\perp$ as $W^+ \oplus W^-$ s.t. $\alpha(W^+) = W^-$ and $W^- = W^+$

$[0,\infty] \to \mathrm{Conf}^{fin}$ $\lambda \longmapsto \; \overset{W^-}{\underset{-\lambda}{\bullet}} \longrightarrow \overset{W_1}{\underset{0}{\bullet}} \longrightarrow \overset{W^+}{\underset{\lambda}{\bullet}} \to$

β

In fact homeomorphism

Fact: $BQ \cong BO \times \mathbb{Z}$

Conj: $1|1\text{-RFT}[X]^n \cong KO^n(X)$

So $1|1\text{-RFT}(X) \simeq$ Chains of $KO(X)$?