$2|1$-EUCLIDEAN FIELD THEORIES AND TOPOLOGICAL MODULAR FORMS

1. Partition functions and moduli stacks

In order to understand TMF and its connection to $2|1$-EFT's, we will use the language of moduli stacks. These have appeared implicitly in the previous weeks, since a fibered category is part of the data of a stack (see Week 3 notes). Given a $d|\delta$-TFT E, its *partition function* will be defined as a function $Z_E : \mathcal{M} \to \mathbb{C}$ from some moduli stack $\mathcal{M}$ associated to $d|\delta$-TFT's to the complex numbers.

1.1. **Partition functions.** Let's start by defining partition functions for d-TFT's. Let E be a d-TFT, so E associates a topological vector space $E(M)$ to every $(d-1)$-dimensional manifold and associates a continuous linear map $E(\Sigma) : E(M_1) \to E(M_2)$ to any d-dimensional bordism $\Sigma : M_1 \to M_2$. If we consider only closed bordisms Σ (i.e. morphisms from the empty manifold to the empty manifold) then $E(\Sigma) : \mathbb{C} \to \mathbb{C}$ is just a complex number.

Let $\mathcal{M}_d$ denote the moduli stack of connected closed d-manifolds. In other words, a map from a space $X \to \mathcal{M}_d$ determines a family of connected closed d-manifolds over X.

Definition 1.1. The *partition function* of E, denoted Z_E, is the function

$$\mathcal{M}_d \to \mathbb{C}$$

defined by

$$\Sigma \mapsto E(\Sigma).$$

These partition functions have played a fundamental role in the previous weeks' results.

Example 1.2. The moduli stack of closed 0-manifolds $\mathcal{M}_0$ consists of a single point (the one-point bordism). Since E is symmetric monoidal, the value of the partition function $Z_E : \mathcal{M}_0 \to \mathbb{C}$ is determined by the value of E on the one-point bordism.

In this case, 0-TFT's are determined by their partition functions since the only (-1)-dimensional manifold is the empty manifold and therefore all morphisms in $0 - Bord$ are closed. When we move to the supersymmetric case, i.e. $0|1$-TFT's, we needed to compute the value of partition functions on superpoints $\mathbb{R}^{0|1}$. It was the identification of these values in Week 2 that allowed us to relate $0|1$-TFT's with the de Rham complex.

The story became more complicated in Week 5 when we discussed $1|1$-EFT's, since there are substantially connected 1-manifolds than connected 0-manifolds.

Example 1.3. The moduli stack $\mathcal{M}_1$ is more complicated than $\mathcal{M}_0$. One simplification we made was to consider the moduli stack $\mathcal{M}_1^E$ of closed *Euclidean* 1-manifolds. In this case, Euclidean just means that everything is equipped with a flat Riemannian metric, so in particular bordisms have lengths. The Euclidean 1-bordisms we were concerned with were intervals of specified length, i.e. bordisms from a point to a point, and we studied the partition functions with domain this moduli stack. Since $E(I_t)$ specified an endomorphism of $E(pt)$, the value of E on any bordism was determined by its value on these intervals plus knowledge of E on elbows. In other words, we identified 1-EFT's by their "partition functions" from the moduli space of intervals. We then considered the supersymmetric case, which imposed certain conditions (skew-adjoint Fredholm) on the eigenvalues and eigenspaces of the operators in $End(V)$. We then identified the space of such operators with $BO \times \mathbb{Z}$ to obtain a relationship between $1|1$-EFT's and K-theory.

As motivation for the definition of TMF and elliptic cohomology, let's consider partition functions for 2|1-EFT's. As in the previous weeks, we begin by studying 2-TFT's. In this case, the moduli stack of closed connected 2-manifolds $\mathcal{M}_2$ is difficult to understand. For example, it contains all genus g surfaces.

We can simplify this picture by restricting to 2-EFT's. Then the moduli stack of closed Euclidean 2-manifolds $\mathcal{M}_2^E$ consists of closed 2-manifolds equipped with a flat Riemannian metric. By the Gauss-Bonnet Theorem, every such 2-manifold is a flat torus. In other words, we only need to consider the values of E on the moduli stack of flat tori $\mathcal{M}_{tori}$. Any such torus can be obtained as the quoteint of $\mathbb{C}$ by a lattice Λ, and any such lattice is determined by a choice of vector $\tau \in \mathbb{H}$ in the upper half-plane by the assignment $\tau \mapsto \mathbb{Z}\{1, \tau\}$.

What happens when we impose supersymmetry? Before discussing this, we need to introduce another moduli stack. An elliptic curve over the complex numbers is the set of solutions to an equation of the form

$$y^2 = x^3 + Ax + B$$

where $A, B \in \mathbb{C}$. It turns out that each elliptic curve is isomorphic (as Riemann surfaces) to a torus $\mathbb{C}/\mathbb{Z}\{1, \tau\}$ where τ is some complex number in the upper half plane $\mathbb{H}$. Two elliptic curves are isomorphic if there exists an element $g \in SL_2(\mathbb{Z})$ such that $\tau = g\tau'$ where $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ via fractional linear transformations.

Definition 1.4. The moduli stack of elliptic curves is the quotient stack

$$\mathcal{M}_{ell} := \mathbb{H}//SL_2(\mathbb{Z}).$$

Any flat torus $T = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z}\{1, \tau\}$ gives rise to an elliptic curve C_τ, so we have a surjection

$$\mathcal{M}_{tori} \twoheadrightarrow \mathcal{M}_{ell}.$$

This map is not an equivalence, since flat tori with different volume (i.e. different choice of τ) often correspond to the same elliptic curve. Nevertheless, the moduli stack of elliptic curves plays an important role in the study of partition functions for 2|1-EFT's.

Theorem 1.5. [?] *Let E be a 2|1-EFT. Then its partition function*

$$Z_E : \mathcal{M}_{tori} \to \mathbb{C}$$

is holomorphic.

In other words, imposing supersymmetry takes us from the study of 2-EFT's to the study of 2-CFT's (conformal field theories). We will see a proof of this theorem next week. Since identifications of the partition functions in the previous weeks led us to de Rham cohomology and K-theory, we have the following question:

Question 1.6. *Is there a cohomology theory related to the moduli stack of elliptic curves?*

2. Elliptic cohomology

In this section, we see that the answer to the question above is a resounding "yes."

2.1. A crash course on formal group laws.

The material in this subsection is mostly taken from Rezk's "Notes on the Hopkins-Miller Theorem" and the TMF book. Let R be a commutative ring. A *formal group law* over R is a formal power series $F(x, y) \in R[[x, y]]$ such that

(1) $F(x, 0) = F(0, x) = x$,
(2) $F(x, y) = F(y, x)$,
(3) $F(F(x, y), z) = F(x, F(y, z))$.

Example 2.1. (1) The *additive formal group law* is defined by $F(x, y) = x + y$.
(2) The *multiplicative formal group law* is defined by $F(x, y) = x + y + xy$.

Definition 2.2. Suppose that R has characteristic p and let F be a formal group law over R. The *p-series* of F is defined by

$$[p]_F(x) = F(F(F(\cdots F(x,x),x),\cdots),x) \text{``}=\text{''} x^{p^n} + ax^{p^n+1} + \cdots .$$

The number n is called the *height* of F.

Exercise 2.3. *Show that the height of the additive formal group law is* ∞ *and the height of the multiplicative formal group law is* 1.

Formal group laws arise naturally in the study of "complex-oriented cohomology theories".

Example 2.4. Using the Atiyah-Hirzebruch spectral sequence, one can show that $H\mathbb{Q}^*(\mathbb{C}P^\infty) \cong H\mathbb{Q}^*[[x]]$ where $|z| = 2$. The map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ classifying the exterior tensor poduct of two copies of the universal line bundle induces a map

$$H\mathbb{Q}^0[[z]] \to H\mathbb{Q}^0[[x,y]].$$

The generator z maps to a formal power series $F_{H\mathbb{Q}}(x,y)$ which is a formal group law over $H\mathbb{Q}^0$ associated to $H\mathbb{Q}$.

Exercise 2.5. *Show that* $F_{H\mathbb{Q}}(x,y)$ *is the additive formal group law over* $\mathbb{Q}$ *and that* $F_{KU}(x,y)$ *is the multiplicative formal group law over* $\mathbb{Z}$.

Another large source of formal group laws are elliptic curves. In the exercise session, we will see that elliptic curves can be equipped with an abelian group structure. It follows that each elliptic curve C over R gives rise to a formal group law $F_C(x,y)$ over R. This formal group law has height 1 if C is an *ordinary* elliptic curve and height 2 if C is a *supersingular* elliptic curve. This defines a map

$$\mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

from the moduli stack of elliptic curves to the moduli stack of formal groups. Moreover, this map is *flat*. We refer the reader to Definition 3.4 of Hohnhold's "The Landweber Exact Functor Theorem" in the TMF book for a precise definition.

2.2. **The Landweber exact functor theorem.** One can define a cohomology theory called complex cobordism MU which associates to a space X the abelian group of complex bordism classes of manifolds over X with a complex linear structure on the stable normal bundle. Evaluating this on a point gives the coefficient ring

$$MU_* \cong \mathbb{Z}[b_1, b_2, \ldots]$$

which by the work of Quillen is isomorphic to the Lazard ring L over which the universal formal group F_u is defined. In other words, if R is a ring, then a ring homomorphism $L \to R$ specifies a formal group over R. Conversely, any formal group over R gives rise to a ring homomorphism $L \to R$. Let $E_*(-) : Top \to Ab$ be defined by

$$E_*(X) := MU_*(X) \otimes_{MU_*} R.$$

For which MU_*-modules R does this assignment define a homology theory? It's clear that this definition is homotopy invariant and satisfies additivity, so the only issue is with excision. One obvious condition on R that makes $E_*(-)$ a homology theory is flatness over MU_*, but in general this is too strong a requirement in practice. The Landweber exact functor theorem gives a more computable and realistic condition.

Theorem 2.6 (Landweber exact functor theorem)**.** *Suppose that for every prime* p*, there are elements* $v_1, v_2, \ldots \in MU_*$ *so that if the sequence* $(p, v_1, v_2, \ldots, v_n)$ *is a regular sequence for* R *for all* p *and* n*. Then* $E_*(-)$ *is a homology theory.*

Equivalently, $E_*(-)$ *is a homology theory if and only if the morphism* $Spec(R) \to \mathcal{M}_{FG}$ *is flat.*

In the statement of the theorem, the elements $v_1, \ldots, v_n$ are defined as the coefficients $v_i := a_{p^i}$ of the p-series of F_u viewed as a formal group law over $\mathbb{Z}_{(p)}$

$$[p]_{F_u}(x) = \sum_{n \geq 1} a_n x^n.$$

We refer the reader to Landweber's original paper for a proof of the first statement, and to the TMF book for a proof of the second statement.

Exercise 2.7. *Show that the additive formal group and the multiplicative formal groups give rise to the homology theories $H\mathbb{Q}$ and KU, respectively.*

We saw above that every elliptic curve over R corresponds to a map $Spec(R) \to \mathcal{M}_{ell}$. If this map is flat, then the composition $Spec(R) \to \mathcal{M}_{ell} \to \mathcal{M}_{FG}$ is flat. In particular, we can apply the Landweber exact functor theorem to obtain an homology theory

$$Ell_*^{C,R}(X) := MU_*(X) \otimes_{MU_*} R.$$

These are called *elliptic cohomology theories.*

3. From elliptic cohomology to TMF

Let's pause to remember why we were interested in elliptic cohomology theories. We had seen that studying partition functions of 0|1-EFT's led us to de Rham cohomology, that partition functions of 1|1-EFT's led us to K-theory, and that partition functions of 2|1-EFT's led us to study functions on the moduli stack of elliptic curves. Unlike in the 0|1- and 1|1-cases, we ended up with a huge class of cohomology theories to choose from (one for each nice elliptic curve). Which elliptic cohomology theory should be related to 2|1-EFT's?

As we saw above, partition functions of 2|1-EFT's don't have a preferred elliptic curve. So, we would like to say that 2|1-EFT's are related to some "universal" elliptic cohomology theory which arises from combining/gluing together all elliptic cohomology theories. The following theorem is due to Goerss-Hopkins-Miller.

Theorem 3.1. *There is a sheaf of E_∞ ring spectra $\mathcal{O}^{top}$ on (the étale site of) the moduli stack of elliptic curves $\mathcal{M}_{ell}$.*

The spectrum of topological modular forms*, denoted TMF, is the global sections of $\mathcal{O}^{top}$.*

Let's try to understand this statement by relating it to what we did in previous weeks. In Week 3, we used the language of fibered categories to describe families of field theories over a generalized supermanifold. Instead of thinking of families of field theories, we want to think of families of elliptic cohomology theories, so we replace the fibered categories $d|\delta - Bord(X)$ by the moduli stack of elliptic curves $\mathcal{M}_{ell}$. We need the additional data of a stack (which is in particular a fibered category) to make sense of sheaves. If we define our Grothendieck topology (covers) nicely, then the assignment above which takes an elliptic curve C defined by a flat map $Spec(R) \to \mathcal{M}_{ell}$ defines a presheaf of elliptic cohomology theories.

We could do this for every ring R, in which case we obtain the Grothendieck site of flat affine schemes over $\mathcal{M}_{ell}$. If this presheaf of elliptic cohomology theories is actually a sheaf, then roughly speaking, we can recover the global sections as a homotopy limit of (families of) elliptic cohomology theories arising from evaluating this sheaf on an open cover, i.e. by gluing together all elliptic cohomology theories.

The phrase "E_∞ ring" in the theorem also has a role when we think about twisted 2|1-EFT's. Implicit in the study of twisted 0|1-EFT's and 1|1-EFT's is multiplicative structure arising from varying how twisted everything is. A similar phenomenon should occur for 2|1-EFT's, and we can ask that the corresponding cohomology theory have a compatible multiplicative structure. We refer the reader to last semester's GSTS notes for precise definitions of E_∞ rings. In the case of TMF, the additional condition E_∞ is used in Goerss-Hopkins obstruction theory to simplify the calculation of certain terms in obstruction theory.

With each phrase in the theorem somewhat motivated, we can now state the main conjecture:
Conjecture 3.2. [?, Conj. 1.17] *There is an isomorphism*

$$2|1 - EFT^n_{loc}[X] \cong TMF^n(X)$$

compatible with the multiplicative structure.

4. Alternative motivation: The Steenrod algebra and stable homotopy theory

Recall that the mod p Steenrod algebra A is defined to be the Hopf algebra (over $\mathbb{F}_p$) of stable cohomology operations for mod p cohomology. When $p = 2$, the Steenrod algebra is generated by the "Steenrod squares" Sq^i with $i \geq 1$ subject to relations called the Adem relations. One can define $A(d)$ to be the subalgebra of A generated by $Sq^1, Sq^2, \ldots, Sq^{2^n}$. For example, we have

$$A(-1) = \{1\},$$
$$A(0) = \{1, Sq^1\},$$
$$A(1) = \{1, Sq^1, Sq^2, Sq^2Sq^1, Sq^3, Sq^2Sq^2, Sq^2Sq^1Sq^2, Sq^3Sq^1, Sq^2Sq^2Sq^2\}.$$

If (Γ, k) is a Hopf algebra and $\Lambda \subset \Gamma$ is a subalgebra of Γ, then we may define the Hopf algebra quotient

$$\Gamma//\Lambda = \Gamma \otimes_\Lambda k.$$

Let $H = H\mathbb{F}_2$ denote mod 2 homology. Then we have

$$H^*(H) = A \cong A//A(-1),$$
$$H^*(H\mathbb{Z}) \cong A//A(0),$$
$$H^*(ko) \cong A//A(1),$$

where ko is the connective version of real topological K-theory and $H\mathbb{Z}$ is the spectrum representing homology with integer coefficients. Naturally, one should ask if there are cohomology theories/spectra with cohomology $A//A(d)$ for all $d \geq -1$. In particular, we see that the Hurewicz image of each cohomology theory increases in the cases above as we increase d, so it would be nice to be able to continue this pattern.

For $d = 2$, we can continue this pattern with a cohomology theory tmf called *(connective) topological modular forms*. In other words, we have

$$H^*(tmf) \cong A//A(2).$$

By the work of Davis-Mahowald, there cannot exist a cohomology theory E with cohomology $H^*(E) \cong A//A(d)$ for $d > 2$, so in a sense tmf is the end of this story.

As we saw above, TMF does not have a geometric interpretation. If the conjecture above is true, then we would be able to make various computations which are currently very difficult, e.g. characteristic classes for TMF.

5. Exercises

(1) Show that every closed flat Riemannian 2-manifold is a flat torus.
(2) Complete the exercises listed in the notes above.
(3) Show that if E is complex orientable, then $E^*(\mathbb{C}P^\infty) \cong E^0[[z]]$ with $|z| = 2$. Relate this to characteristic classes for E-cohomology.
(4) This exercise describes how to define the group law on an elliptic curve. For now, let's work over $\mathbb{C}$. Fix an origin O on C and let $P, Q \in C$ be two other points. Then $P \oplus Q$ is defined as follows. Let $P * Q$ denote the unique third point of intersection of the line passing through P and Q and the elliptic curve C. Then $P \oplus Q$ is the unique third point of intersection of the line passing through $P * Q$ and O and the elliptic curve C. Draw a picture of an elliptic curve over R and convince yourself through pictures (or the Riemann-Rock theorem) that this is a group law.

(5) This exercise describes how to obtain a formal group law from an elliptic curve. We refer the reader to David Loeffler's lecture notes on Elliptic Curves for details.

(a) In the affine coordinates $z = -x/y$ and $w = -1/y$, any elliptic curve C has generalized Weierstrass equation

$$w = z^3 + (a_1 z + a_2 z^2)w + (a_3 + a_4 z)w^2 + a_6(w^3) =: f(z, w).$$

The origin here is at $w = 0$. We can think of w as a function of z by repeatedly plugging $f(z, w)$ in for w above. The resulting power series is called $w(z)$.
Show that $w(z)$ has the form

$$w(z) = z^3 + a_1 z^4 + (a_1^2 + a_2)z^5 + (a_1 z^3 + 2a_1 a_2 + a_3)z^6 + \cdots .$$

(b) Using this power series, we can obtain points in in a neighborhood of the origin as $(w(z), z)$. If we have points (w_1, z_1) and (w_2, z_2) in this chart (where here we think of z_1, z_2 as variables), let λ denote the slope of the line passing through them, and let $c = w_1 - \lambda z_1$, so we have $w = \lambda z + c$. Substituting this into the Weierstrass equation, we get a cubic in z with solutions. Show that the inverse of the sum of z_1 and z_2 is given by

$$i(F(z_1, z_2)) = \frac{a_1 z + a_2 c + a_3 \lambda^2 + 2a_4 \lambda c + 3a_6 \lambda^2 c}{1 + a_2 \lambda + a_4 \lambda^2 + a_6 \lambda^3} - z_1 - z_2.$$

(c) Conclude that

$$F(z_1, z_2) = -i(F(i(F(z_1, z_2)), 0))$$

is a formal group law.