EXTENDED TQFT'S AND HIGHER CATEGORY THEORY

TIM CAMPION

1. Higher category theory

Recall that an *functorial field theory* is a symmetric monoidal functor

$$Z : d - Bord \to \mathcal{C}.$$

This allows us to calculate:

[picture of a surface decomposed into pairs of pants]

This is great for 2-dimensional manifolds, but gets very complicated even for 3-dimensional manifolds. For more details on the 3-dimensional case, these are Heegaard splittings (which we won't discuss here).

In general, we'd like to keep cutting our manifolds further. To make this notion precise, we will use *higher category theory*.

We begin with some examples.

Example 1.1. The category Cat of categories is a 2-category, where objects are categories, morphisms are functors, and 2-morphisms are natural transformations.

Example 1.2. Let $\pi_{\geq 2}(X)$ be the fundamental 2-groupoid of a topological space X. The objects are points in X, the morphisms are paths in X, and the 2-morphisms are homotopies between paths.

The previous example generalizes and suggests the following definition.

Definition 1.3. A *strict n-category* is a category $\mathcal{C}$ enriched in $(n-1)$-categories. A 0-category is a set.

2. Extended bordism categories

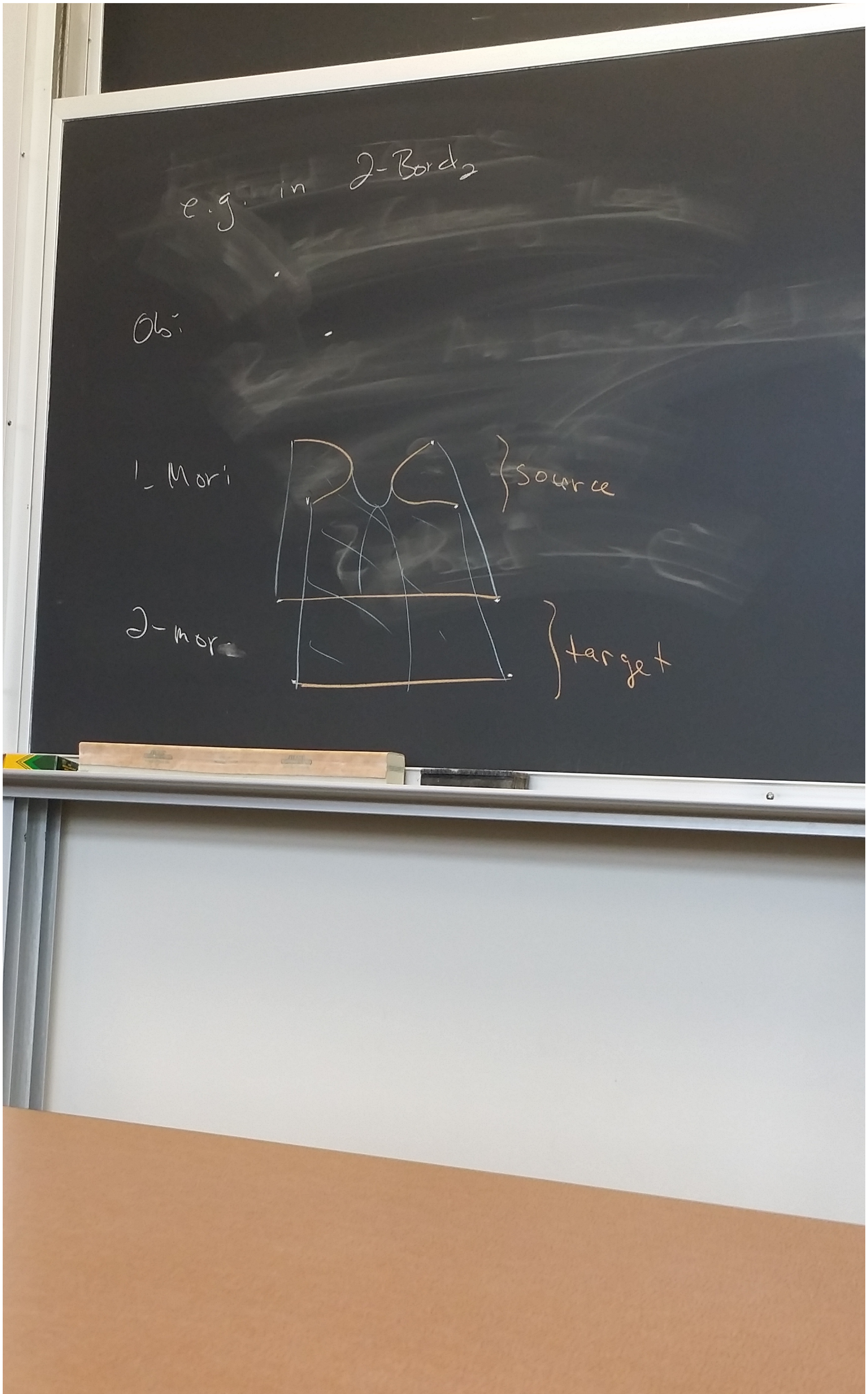
Our goal is to define an n-category of n-cobordisms.

Definition 2.1 (Sketch Definition)**.** Let $n \leq d$ and let X be some structure group. Define the *extended bordism category* $d - Bord_n^X$ to be the n-category where the objects are $(d-n)$-dimensional closed manifolds with structure X, the 1-morphisms are X-bordisms between objects, the 2-morphisms are X-bordisms between X-bordims, and so on until we get to n-morphisms.

When $n = d$, this is a "fully extended" bordism category.

Let's consider a specific example.

Example 2.2. Let's study $2 - Bord_2$. The objects are collections of points, the 1-morphisms are bordisms between these points, i.e. curves with boundary the points, and the 2-morphisms are bordisms between them. The following picture illustrates the end result.


e.g. in 2-Bord$_2$
Ob:
1-Mor:
2-mor
source
target

Recall from earlier in the semester that an oriented 1-dimensional TFT

$$Z : 1 - Bord^{or} \to Vect_{\mathbb{C}}$$

is determined by its values on the positively oriented point and the negatively oriented point. These could not go to just any vector spaces; studying the values on bordisms between these gave us maps

$$V \otimes W \xrightarrow{\epsilon} \mathbb{C},$$

$$\mathbb{C} \xrightarrow{\eta} W \otimes V.$$

Evaluating Z on certain 1-bordisms (snakes) shows us that the composite

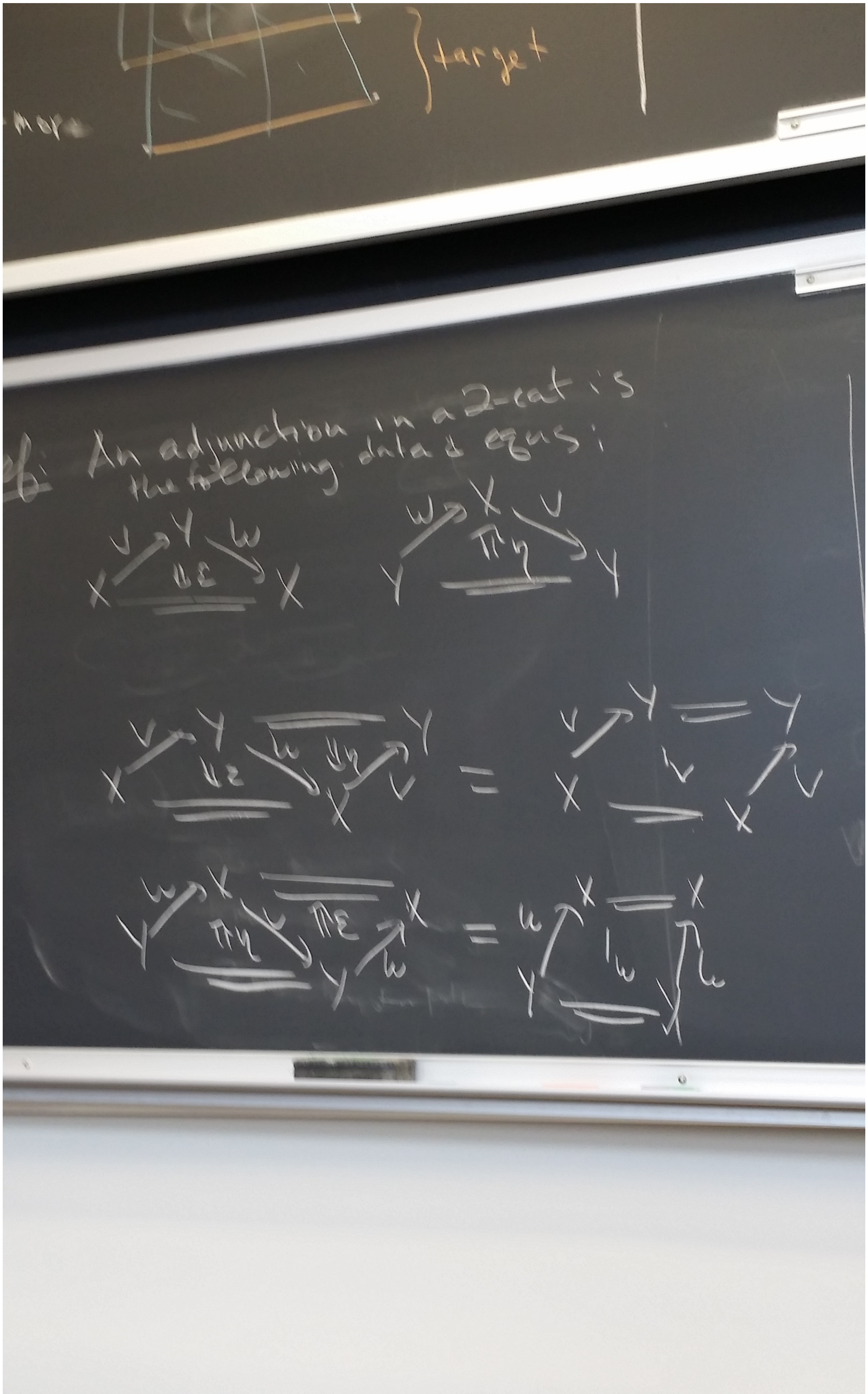
$$V \to V \otimes W \otimes V \to V$$

is equal to the identity on V. Similarly, one can obtain something similar for W.

We conclude that we must have $W = V^{\vee}$, that ϵ is the pairing and η is the copairing. This structure is called a *dual pair*, and it makes sense in any symmetric monoidal category! It turns out that $(+, -)$ is a dual pair in the category $1 - Bord^{or}$, so it must be sent to a dual pair in $Vect$. It follows that $1 - Bord^{or}$ is the free symmetric category on an object with a dual.

This idea extends further.

Definition 2.3. An *adjunction* in a 2-category is the following data and equations.


target
An adjunction in a 2-cat is
the following data & eqns:

Exercise 2.4. *For all symmetric monoidal categories $\mathcal{C}$, there exists a 2-category $B\mathcal{C}$ such that a dual pair in $\mathcal{C}$ is an adjunction in $B\mathcal{C}$.*

Now let's consider another example.

Example 2.5. Consider the 2-category $2-Bord_2^{fr}$, where the superscript "fr" tells us that we've specified a framing at each point. The unit η and counit ϵ of the adjunction themselves have adjoints.

[[picture of morphism from counit and counit dual down to the identity is the same as colored picture before]]

Exercise 2.6. *Check that the equations for an adjunction in a 2-category are satisfied.*

Exercise 2.7 (Exercise for the ambitious)**.** *Convince yourself using Morse theory that this happens at all dimensions, where in higher dimensions the unit and counits are given by handles. That is, units and counits are disks going from $S^k \times D^{d-k-1}$ to $D^{k+1} \times S^{d-k}$.*

We can now state the main theorem.

Theorem 2.8 (Cobordism Hypothesis)**.** *For all d, the fully extended bordism d-category $d-Bord_d$ satisfies the following universal property:*

$$Fun^{\otimes}(d-Bord_d^{fr},\mathcal{C}) \cong \mathcal{C}^{fd}$$

where $\mathcal{C}^{fd} \subset \mathcal{C}$ is the largest symmetric monoidal d-subcategory where every object has a dual and every k-morphism (for every $k \leq d-1$) has left and right adjoints.

One can try to understand what happens if we replace "framed" by other geometries.

Example 2.9. A functor $F \in Fun^{\otimes}(1-Bord^{unor},\mathcal{C})$ corresponds to a vector space V with a specified isomorphism to its dual, i.e. a choice of bilinear form on V.

3. Unweaving the web of lies

It turns out that these n-categories don't exist.

Example 3.1. Recall that we defined $\pi_{\leq n}(X)$ before. It turns out that this is not actually an n-category. The problem is that we have to composition by specificying a parametrization of our path, but then there are issues with associativity. For the fundamental groupoid $\pi_{\leq 1}(X)$, we can get away with taking homotopy classes of paths. However, for $\pi_{\leq 2}(X)$, if you take homotopy classes too soon there are issues.

To rectify this problem, we will need *weak* higher categories. As a motivating example, we do not have an equality between $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$. Instead we have an isomorphism. If we go up to four objects, we have to study the usual pentagon diagram to keep track of coherence, and things get worse as we add more terms.

One thing we didn't talk about before was how to identify the top-dimensional bordisms. Implicitly we assumed that they were identified up to diffeomorphism, but it turns out that this isn't the best thing to do. Instead, in order to define a weak higher category, it's easier to define an ∞-category where morphisms above a certain level are invertible.

The reason is the *Homotopy Hypothesis*. Consider $\pi_\infty(X)$. This is an $(\infty,0)$-category and $Sing(X)$ is a model for it. It turns out that there is an embedding

$$(\infty,n)-Cat \hookrightarrow [\Delta^{op} \times \cdots \times \Delta^{op}, Top],$$

$$(k_1,\ldots,k_n) \mapsto \{ \text{ morphisms subdivied into } n\text{-cubes with all higher morphisms}\}.$$

We actually want to think of these as simplicial spaces.

Definition 3.2. Define $d - Bord_n$ to be the n-fold simplicial space

$$\Delta^{op} \times \cdots \times \Delta^{op} \to Top$$

which sends $(k_1, \ldots, k_n)$ to the space of d-manifolds with corners equipped with subdivisions into a grid.