CLASSIFYING INVERTIBLE TFT'S VIA COHOMOLOGY

TJ WARNER

1. Invertible TFT's

Recall that last week Tim defined (∞, n)-categories which are certain n-fold simplicial spaces

$$\mathcal{C} : (\Delta^{op})^n \to Top.$$

He defined the (∞, n)-category of d-bordisms $d - Bord_n$, and defined TFT's as symmetric monoidal functors

$$E : d - Bord_n \to \mathcal{C}$$

where $\mathcal{C}$ is some symmetric monoidal category.

Example 1.1. Recall the $(\infty, 1)$-category of oriented 2-bordisms, denoted $2 - bord_1^{SO}$. We showed in teh first exercise session that any symmetric monoidal functor from there to $Vect_1$ corresponds to a commutative Frobenius algebra over $\mathbb{C}$.

Exercise 1.2. *What is $Vect_1$ as a functor $\Delta^{op} \to Top$?*

Definition 1.3. An *invertible TFT* is a TFT which sends objects to $\otimes$-invertible objects and sends morphisms to invertible morphisms.

We want to define a target category where the only possible targets are invertible.

Definition 1.4. An ∞-*Picard category* is one in which all objects are $\otimes$-invertible and all morphisms are invertible.

Define $Pic(\mathcal{C})$ to be the maximal ∞-Picard subcategory.

In other words, an invertible TFT is a functor which factors through $Pic(\mathcal{C})$. From now on, we will denote Picard categories by E.

Exercise 1.5. *Show that E is an ∞-Picard category if and only if the functor*

$$(\otimes, proj_1) : E \times E \to E \times E$$

is an equivalence.

We saw above that oriented 2-dimensional TFT's correspond to commutative Frobenius algebras over $\mathbb{C}$. We want to know which of these are invertible. To figure this out, we observe that $Pic(Vect_1)$ has a single object $\mathbb{C}$ and has morphisms $\mathbb{C}^\times$. In other words, invertible oriented 2-TFT's correspond to 1-dimensional commutative Frobenius algebras.

[[picture of unit, pair of pants, upside down pair of pants, and counit from last week]]

The bordisms above correspond to the structure maps in a Frobenius algebra.

Exercise 1.6. *An invertible oriented 2-dimensional TFT Z_1 satisfies $Z_1(\Sigma_g) = \mu^{1-g}$ where $\mu \in \mathbb{C}^\times$ is the value*

Everything we have discussed so far only relied on the 1-categorical structure.

Definition 1.7. Define a 2-category $Vect_2$ with objects natural numbers, morphisms $m \times n$ matrices of vector spaces, and 2-morphisms $m \times n$ matrices of linear maps. Composition of 1 -morphisms is given by matrix multiplication where we use direct sum and tensor product.

$Vect_2$ is a categorical delooping of $Vect_1$ in the sense that

$$Hom(1,1) = Vect_1.$$

Furthermore, we have $Pic(Vect_2)$ has a single object 1, morphisms 1-dimensional vector spaces, and 2-morphisms $\mathbb{C}^\times$. But then the category with objects the 1-morphisms of $Pic(Vect_2)$ and morphisms the 2-morphisms of $Pic(Vect_2)$ is just $Pic(Vect_1)$!

More generally, weh ave the following definition.

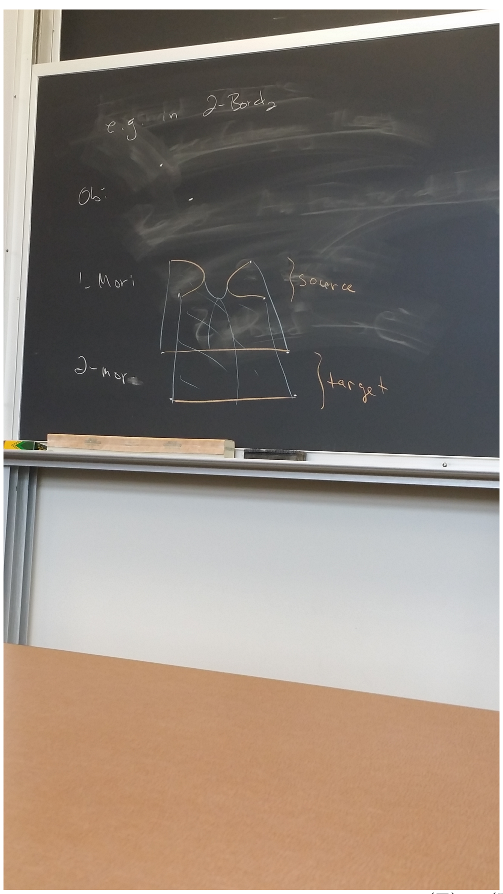
Definition 1.8. If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category, then define $B\mathcal{C}$ to be the $(\infty, n+1)$-category with one object $*$ and $Hom(*, *) = \mathcal{C}$.

Example 1.9. We have $B(Pic(Vect_1)) \simeq Pic(Vect_2)$.

We now want to understand the diagram

$$\begin{array}{ccc} Z_2 : 2 - Bord_2^{SO} & \longrightarrow & Vect_2 \\ \downarrow & \nearrow & \\ Pic(Vect_2). & & \end{array}$$

We have already specified where the objects and 1-morphisms are sent, so we just need to understand 2-morphisms. These can be determined by studying the following 2-morphism (see last week's notes for a bigger version):



We pick $\lambda \in \mathbb{C}^\times$ and set $Z_2(\Sigma) = \lambda^{\chi(\Sigma) - \chi(Y)}$.

Exercise 1.10. *This defines a TFT with*

$$Z_2(\Sigma_2) = \lambda^{\chi(\Sigma_g)} = \lambda^{2-2g} = (\lambda^2)^{1-g}.$$

Note that $2 - Bord_1$ sits inside of $2 - Bord_2$, but to make this a map in categories, we need to take

$$B(2 - Bord_1) \to 2 - Bord_2$$

where the left-hand side sits inside as the morphisms of the empty manifold. Examining the variance of the above map shows that we obtain a map

$$2 - TQFT_2^{inv} \to 2 - TQFT_1^{inv}.$$

At the level of commutative Frobenius algebras, this sends the number λ to the number λ^2. In other words, specifying further information via higher categories allows us to distinguish between λ and $-\lambda$.

We can inductively define higher categories $Vect_n$ with $Pic(Vect_n)$ having one thing in all levels, except in the n-th level we have $\mathbb{C}^\times$. We think of $Pic(Vect_n)$ as a model for $K(\mathbb{C}^\times, n)$.

2. Classifying invertible TFT's

For our (∞, n)-categories, "morphisms being invertible" corresponds to "being equivalent to a constant multisimplicial space." This in turn is equivalent to being an $(\infty, 0)$-category above level n. Therefore we have the following key idea: Picard (∞, n)-categories model connective spectra. In particular, objects being invertible gives the group-like condition on the 0-space, i.e. gives the structure of a group-like space (i.e. infinite loop space) and therefore a connective spectrum.

For E Picard, if we have a map

$$d - Bord_n \to E,$$

we can pass to geometric realization

$$||d - Bord_n|| \to E.$$

Such maps then correspond to classes in

$$\pi_0 Map_{E_\infty}(||d - Bord_n||, E)$$

which in turn correspond to classes in the E-cohomology of the spectrum associated to $||d - Bord_n||$. Note that above $|| - ||$ is the fat realizaiton which can be thought of as taking sequential geometric realization with respect to each simplicial direction.

Remark 2.1. *Although $d - Bord_n$ is not grouplike, it becomes grouplike after taking geometric realization. In other words, this realization forces disjoint union to become invertible.*

3. Madsen-Tillman spectra

From now on, we want to understand $d - Bord_n^{SO(d)}$. We begin by defining Madsen-Tillman spectra. Let γ_d be the tautological d-plane bundle and let $\gamma_d^\perp = \epsilon_{p+1} - \gamma_d$ be its complement. Consider the diagram

$$\begin{array}{ccc} \gamma_d^\perp & & \gamma_d \\ \downarrow & & \downarrow \\ Gr_d(\mathbb{R}^p) & \xrightarrow{i} & Gr_d(\mathbb{R}^{p+1}). \end{array}$$

We have $i^*(\gamma_d^\perp) \cong \gamma_d^\perp \oplus \epsilon$, so taking Thom spaces we obtain maps

$$\Sigma(Th(\gamma_d^\perp)) \overset{\cong}{\to} Th(i^*\gamma_d^\perp).$$

This defines a spectrum $MTSO(d)$ called the *Madsen-Tillman spectrum* defined by setting

$$(MTSO(d))_p = Th(\gamma_d^{p\perp}).$$

Remark 3.1. *The 'M' above is the classical notation for Thom spectra, and the 'T' is to remind us that there are tangential structures floating around.*

These spectra are usually $(-d)$-connective.

Theorem 3.2 (Galatius-Madsen-Tillman-Weiss, Schommer-Pries)**.** *There is an equivalence of E_∞-spaces*

$$||d - Bord_n|| \simeq \Omega^{\infty - n} MTSO(d) = \Omega^\infty \Sigma^n MTSO(d).$$

We now have the spectrum associated to $||d - Bord_n||$. We need to compute its homotopy or cohomology.

Example 3.3. For $k < d$, we have a group isomorphism

$$\pi_k \Sigma^d MTSO(d) \cong \Omega_k^{or}.$$

We now want to compute cohomology with integer coefficients. Using the short exact sequence of groups

$$\mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times$$

we can recover the cohomology with coefficients in $\mathbb{C}^\times$ which relates us back to $Pic(Vect_n)$. We find that

$$d - TQFT_n^{inv} \cong H^d(p_{\geq d-n} \Sigma^d MTSO(d); \mathbb{C}^\times)$$

where $p_{\geq d-n}$ is the Postnikov cover which throws away homotopy groups below degree $d - n$.

It is also interesting to contemplate the comparison maps

$$B(d - Bord_n) \longrightarrow d - Bord_{n+1}$$

$$\begin{array}{ccc} 2 - TQTF_1^{inv} & \longleftarrow & 2 - TQFT_2^{inv} \\ & & \downarrow \\ & & 1 - TQFT_1^{inv} \end{array}$$

$$d - Bord_n \longrightarrow (d+1) Bord_{n+1}$$

Example 3.4.
- For $d = 1, 3$ and for any $n \leq d$, there is a unique TQFT.
- For $d = 2$, there is a different TQFT for each element of $\mathbb{C}^\times$.
- For $d = 4$, the different TQFT's are indexed by pairs of complex numbers in $\mathbb{C}^\times \times \mathbb{C}^\times$.

The restriction maps are bijective except for

$$4 - TQFT_4^{inv} \to 4 - TQFT_3^{inv}$$

which is 6-to-1.

Another fun fact:

$$\Sigma MTSO(1) \simeq S^0.$$