# Spring 2019 Graduate Student Topology Seminar
# Functor Calculus

Jens Jakob Kjaer

January 11, 2019

Functor calculus is an attempt to, by analogy, treat functors as smooth functions. The primary method is approximating a functor of interest by simpler "polynomial" functors. Usually the approach is one of the following three:

1. **Homotopy Calculus** concerns itself with functors of topological spaces and spectra. A functor being linear means that it behaves like a homology theory. The canonical example is the identity functor on topological spaces.

2. **Embedding Calculus** (sometimes called Goodwillie-Weiss calculus) starts with a manifold $M$ and then concerns itself with functors $F : \mathcal{O}(M)^{op} \to Top$, and usually trying to understand $F(M)$. Here a linear functor is a functor that is uniquely defined by what it does to open embedded discs. The canonical example is $F = \mathrm{Emb}(\ , N)$ for some other manifold $N$.

3. **Orthogonal Calculus** (sometimes called Weiss calculus) Concerns itself with functors $F : Vect \to Top$, where $Vect$ is the category of finite dimensional vector spaces with a positive definite inner product. Here the canonical example is the functor $V \mapsto BO(V)$.

In the mini-course I will give an introduction to homotopy calculus, as that is what I am most knowledgeable about. Hopefully the other variations will be covered by participant talks. This unfortunately also means that most of the resources I can provide are in this direction.

## Some resources

### Introductions

Good introductions to both homotopy and embedding calculus can be found in the note set on the topic form Oberwolfach (`www.mfo.de/document/0414/OWR_2004_17.pdf`), as well as the notes from the 2012 Talbot Workshop (`https://math.mit.edu/conferences/talbot/index.php?year=2012&sub=talks`).

A good introduction to orthogonal calculus can be found on Weiss' website: `https://ivv5hpp.uni-muenster.de/u/mweis_02/preprints/or.pdf`

It is also worth reading Goodwillie's original Calculus I-III papers. Calc3 can be found here: `hopf.math.purdue.edu/Goodwillie/calculus3.pdf`. Calc 1 and 2 are unfortunately hard to find, but I can email them if interested.

### The Derivatives of the Identity

The identity functor on pointed spaces is one of the most important functors from the point of view of Homotopy Calculus. It is therefore of great interest to give good description of its derivatives. This was first completed by Johnson (`https://www.ams.org/journals/tran/1995-347-04/S0002-9947-1995-1297532-6/`). A different description better in line with the chain rule topic below was given by Ching (`mching.people.amherst.edu/Work/bar-constructions.pdf`).

### Chain rule

In homotopy calculus one question is how the theory interact with composition of functors. This was answered by Arone and Ching (`https://mching.people.amherst.edu/Work/chain-rule-spaces-final.pdf`) relying on Koszul duality of operads. A more self contained answer that is probably easier to give a talk on is Yeakel's work in the case of functors of spaces (`https://arxiv.org/abs/1706.06915`).

### Chromatic Homotopy theory

The Taylor tower of the identity on topological spaces is closely tied to chromatic homotopy theory. The fundamental computation by Arone and Mahowald can be found here `http://hopf.math.purdue.edu/Arone-Mahowald/`

ArMahowald.pdf, and a survey of the topic by Kuhn here http://arxiv.org/abs/math/0410342.

### Higher Category Theory

Lurie has in Higher Algebra a section on Goodwillie calculus in the context of quasi-categories (section 7). An interesting paper is Heuts' Goodwillie calculus of categories (http://arxiv.org/abs/1510.03304). A different paper on model category theory is Pereira determining exactly what you need to set up functor calculus (https://arxiv.org/abs/1301.2832).

### $K$-theory and $THH$

One of the original motivation of Goodwillie was explaining the relationship between Waldhausen $K$-theory of spaces and THH, this can be found in Calc3, section 9.