MINICOURSE PART I

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1. Introduction

I'll start with an overview, then tell you about homotopy functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We wish to understand $F(X)$ for some preferred object $X \in \mathcal{C}$. By "preferred," we mean that there is a preferred way of constructing objects in $\mathcal{C}$.

Example 1.1. If $\mathcal{C}$ is the category of topological spaces, then the preferred objects are CW complexes.

We also have a preferred way of computing in $\mathcal{D}$.

Example 1.2. If $\mathcal{D}$ is also topological spaces, then we want to compute homology. We then need cofiber sequences. Or, we want to compute homotopy groups, in which case we need fiber sequences.

A very nice functor should take our preferred means of construction to our preferred means of computation. We'll see that most functors are not very nice, but we can filter F so that the layers of F *are* very nice.

We will usually work with $\mathcal{D} = Top_*$ or $\mathcal{D} = Sp$. In these cases, there are special names for functor calculus depending on $\mathcal{C}$:

(1) If $\mathcal{C} = Top_*$, then the area of study is *homotopy calculus*.
(2) If $\mathcal{C} = \mathcal{O}(M)$ is the category of open sets on a manifold, then this is *embedding calculus*. Usually, we have $F = Emb(-, M)$.
(3) There are other cases we're interested in, but these are the ones we will focus on here.

2. Homotopy calculus

Suppose we have $F : Top_* \to Top_*$. Our goal is to compute $\pi_* F(X)$. We make the following assumptions on F:

Definition 2.1. (1) We say that F is *homotopical* if $X \simeq Y$ implies that $F(X) \simeq F(Y)$.
(2) We say that F is *pointed* if $F \simeq *$ implies that $F(X) \simeq *$. Sometimes we will require $F(X) = *$ to make things easier.
(3) We say that F is *finitary* or *continuous* if for any CW complex X, we have

$$F(X) \simeq \operatorname{colim}_{A \subseteq X} F(A)$$

where the colimit is taken over finite CW complexes $A \subseteq X$.

Remark 2.2. *The analog of "pointed" in classical calculus is that $f(0) = 0$. Just as in classical calculus and Taylor series, one can work out homotopy calculus without assuming pointedness. However, the assumption will make things cleaner in the sequel.*

2.1. **Linear functors.**

Example 2.3. Let's take an easy example. Let X be a CW complex. Let $F = \Omega^\infty(H\mathbb{Z} \wedge -)$, so $\pi_* F(X) = \tilde{H}_*(X)$. This is finitary since H is a homology theory. Filter X by its CW decomposition

$$X \supset \cdots \supset X^{[n]} \supset \cdots \supset X^{[0]}.$$

Applying this functor to the filtration quotients gives a spectral sequence (the Atiyah-Hirzebruch spectral sequence)

$$E_1 = \bigoplus_{\text{cells of } X} \pi_* F(S^i) \cong \bigoplus_{\text{cells of } X} H_*(S^i) \Rightarrow \tilde{H}_*(X).$$

This is also just the cellualr chain complex.

We can generalize this example as follows.

Example 2.4. Suppose that X is a space equipped with a tower of Serre fbrations

$$X \to \cdots \to X_\ell \to X_{\ell-1} \to \cdots \to X_0$$

with fibers Y_ℓ. Let $F = id$. Then we obtain a spectral sequences

$$\prod_\ell \pi_* Y_\ell \Rightarrow \pi_* X.$$

We could apply this to a Postnikov tower for X, but this gives a fairly dumb spectral sequence since we input $\pi_* X$ to compute $\pi_* X$. In general, it's rare to be handed X along with a tower of fibrations as above.

Definition 2.5. We say that F is *linear* or 1-*excisive* if it takes homotopy pushouts to homotopy pullbacks.

Remark 2.6. *This definition is motivated by how we build CW complexes. We have homotopy pushouts*

$$\begin{array}{ccc} \bigvee S^{i-1} & \longrightarrow & X^{[i-1]} \\ \downarrow & & \downarrow \\ \bigvee D^i & \longrightarrow & X^{[i]}. \end{array}$$

If F is linear, then we have a homotopy pullback

$$\begin{array}{ccc} F(\bigvee S^{i-1}) & \longrightarrow & F(X^{[i-1]}) \\ \downarrow & & \downarrow \\ F(\bigvee D^i) & \longrightarrow & F(X^{[i]}). \end{array}$$

The bottom-left is a point (since F is pointed) and the top-left is a product of $F(S^{i-1})$.

The remark implies the following.

Proposition 2.7. *Suppose that F is linear and X is a CW complex. Then there is a spectral sequence*

$$\prod_{\text{cells of } X} \pi_* F(S^i) \Rightarrow \pi_* F(X).$$

This spectral sequence does not always converge, but with sufficient connectivity and finiteness conditions it does converge.

Example 2.8. We've seen $F = \Omega^\infty H\mathbb{Z} \wedge -$ above. We can generalize this by taking $F = \Omega^\infty(C \wedge -)$ where C is any spectrum. In fact, these are the only examples! This follows from Brown representability and unwinding the definition of linearity; we will work this out in detail later.

Assume that F is linear. We can apply F to the pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

to obtain a homotopy pullback

$$\begin{array}{ccc} F(X) & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & F(\Sigma X) \end{array}$$

which implies that we have an equivalence

$$F(X) \xrightarrow{\simeq} \Omega F(\Sigma X).$$

Iterating this process, we see that if F is linear, then $F(X)$ is an infinite loop space! This justifies the Ω^∞ in the example above.

Let $\mathcal{L}$ be the category of linear functors from Top_* to Top_*. To any object of $\mathcal{L}$, we can attach a spectrum to obtain a functor

$$\mathcal{L} \xrightarrow{\partial_1} Sp,$$
$$F \mapsto \partial_1 F$$

where

$$(\partial_1 F)_n = F(S^n).$$

The structure maps were described above, and we see that in fact $\partial_1 F$ is an Ω-spectrum.

We can go the other way by setting

$$Sp \to \mathcal{L},$$
$$C \mapsto \Omega^\infty(C \wedge -).$$

Theorem 2.9. *This gives an equivalence of categories between $\mathcal{L}$ and Sp.*

Remark 2.10. *We will not give the proof in detail since we don't want to discuss the model structure on $\mathcal{L}$.*

Why should we believe this? If F is linear, then $F \simeq \Omega^\infty(\partial_1 F \wedge -)$. Then the spectral sequence

$$\prod_{cells} \pi_* F(S^i) \Rightarrow \pi_* F(X)$$

strongly suggests that the theorem is true.

2.2. **Not linear functors?**

We can now study the composite

$$P_1 : Fun(Top_*, Top_*) \xrightarrow{\partial_1} Sp \to \mathcal{L}$$

and the suspension diagram from before gives a map (after resolving point-set issues)

$$F(X) \to \Omega F(\Sigma X).$$

If F is not linear, this may not be an equivalence. In general, we have

$$P_1 F \simeq \Omega^\infty(\partial_1 F \wedge -).$$

This is the *first polynomial approximation to F*. We will discuss the generalization of this next week.

2.3. **Connectivity issues.**

Definition 2.11. A functor F is $E(c, \kappa)$ if given any diagram

$$\begin{array}{ccc} X & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array}$$

where $k_i \geq \kappa$ and $X \to X_i$ is k_i-connected, then the map from $F(X)$ to the homotopy limit of

$$\begin{array}{ccc} & & F(X_1) \\ & & \downarrow \\ F(X_2) & \longrightarrow & F(X_{12}) \end{array}$$

is $(k_1 + k_2 - c)$-connected.

If F is $E(c, \kappa)$ for some c and κ, we say that F is *stably linear*.

Example 2.12. If F is linear, then F satisfies $E(-\infty, -1)$.

Example 2.13. If F is the identity, then F satisfies $E(1, \kappa)$ for any κ. This is just a restatement of the Blakers-Massey Theorem.

Definition 2.14. Let $T_1(F)(X)$ be the homotopy limit of

$$\begin{array}{ccc} & & F(CX) \\ & & \downarrow \\ F(CX) & \longrightarrow & F(\Sigma X). \end{array}$$

Note that $T_1(F)(X) \simeq \Omega F(\Sigma X)$.

Proposition 2.15. *If F is $E(c, \kappa)$, then T, F is $E(c-1, \kappa-1)$.*

Proposition 2.16. *If F is stably linear, then*

$$\operatorname{colim}(F \to T_1(F) \to T_1^{\circ 2}(F) \to \cdots)$$

is linear and agrees with P_1F as defined above.

Remark 2.17. *The (homotopy) colimit above is defined pointwise on CW complexes. This is one place the finitary condition on F is necessary.*

We'll generalize this definition and proposition next week, as well.

Proposition 2.18. *The map $F \to P_1F$ is initial (in the homotopy category of functors) amongst linear functors under F. Equivalently, any map from F to a linear functor factors through P_1F.*

Proof. Let $\mathcal{L}$ be a linear functor and let $F \to L$ be a map. Then consider

$$\begin{array}{ccc} F & \longrightarrow & L \\ \downarrow & & \downarrow \\ T_1(F) & \longrightarrow & T_1(L) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ P_1F & \longrightarrow & P_1(L). \end{array}$$

Everything in the right-hand column is equivalent to L, so we're done. □

Proposition 2.19. *We have*

$$P_1(id) \simeq \Omega^\infty \Sigma^\infty.$$

Proof. We have $T_1(id) \simeq \Omega\Sigma$, and more generally $T_1^{\circ n} \simeq \Omega^n \Sigma^n$. The result follows by letting n tend to infinite. □

Corollary 2.20. *The Hurewicz homomorphism always exists.*

Example 2.21. We have $\partial_1(id) \simeq \mathbb{S}$.

Proposition 2.22. *Let K be a finite CW complex. Then*

$$P_1 Map(K,-) \simeq Map_{Sp}(\Sigma^\infty K, \Sigma^\infty -).$$

In particular,

$$\partial_1 Map(K,-) \simeq DK$$

where D is the Spanier-Whitehead dual functor.