MINICOURSE PART I

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1. INTRODUCTION

I'll start with an overview, then tell you about homotopy functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. We wish to understand F(X) for some preferred object $X \in \mathcal{C}$. By "preferred," we mean that there is a preferred way of constructing objects in \mathcal{C} .

Example 1.1. If C is the category of topological spaces, then the preferred objects are CW complexes.

We also have a preferred way of computing in \mathcal{D} .

Example 1.2. If \mathcal{D} is also topological spaces, then we want to compute homology. We then need cofiber sequences. Or, we want to compute homotopy groups, in which case we need fiber sequences.

A very nice functor should take our preferred means of construction to our preferred means of computation. We'll see that most functors are not very nice, but we can filter F so that the layers of F are very nice.

We will usually work with $\mathcal{D} = Top_*$ or $\mathcal{D} = Sp$. In these cases, there are special names for functor calculus depending on \mathcal{C} :

- (1) If $\mathcal{C} = Top_*$, then the area of study is homotopy calculus.
- (2) If C = O(M) is the category of open sets on a manifold, then this is *embedding* calculus. Usually, we have F = Emb(-, M).
- (3) There are other cases we're interested in, but these are the ones we will focus on here.

2. Homotopy calculus

Suppose we have $F: Top_* \to Top_*$. Our goal is to compute $\pi_*F(X)$. We make the following assumptions on F:

Definition 2.1. (1) We say that F is *homotopical* if $X \simeq Y$ implies that $F(X) \simeq F(Y)$.

- (2) We say that F is *pointed* if $F \simeq *$ implies that $F(X) \simeq *$. Sometimes we will require F(X) = * to make things easier.
- (3) We say that F is *finitary* or *continuous* if for any CW complex X, we have

$$F(X) \simeq \operatorname{colim}_{A \subseteq X} F(A)$$

where the colimit is taken over finite CW complexes $A \subseteq X$.

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Remark 2.2. The analog of "pointed" in classical calculus is that f(0) = 0. Just as in classical calculus and Taylor series, one can work out homotopy calculus without assuming pointedness. However, the assumption will make things cleaner in the sequel.

2.1. Linear functors.

Example 2.3. Let's take an easy example. Let X be a CW complex. Let $F = \Omega^{\infty}(H\mathbb{Z} \wedge -)$, so $\pi_*F(X) = \tilde{H}_*(X)$. This is finitary since H is a homology theory. Filter X by its CW decomposition

$$X \supset \dots \supset X^{[n]} \supset \dots \supset X^{[0]}.$$

Applying this functor to the filtration quotients gives a spectral sequence (the Atiyah-Hirzebruch spectral sequence)

$$E_1 = \bigoplus_{\text{cells of } X} \pi_* F(S^i) \cong \bigoplus_{\text{cells of } X} H_*(S^i) \Rightarrow \tilde{H}_*(X).$$

This is also just the cellualr chain complex.

We can generalize this example as follows.

Example 2.4. Suppose that X is a space equipped with a tower of Serre for for the space of Serre for the space of Sere for the space of Sere for the space of

$$X \to \dots \to X_{\ell} \to X_{\ell-1} \to \dots \to X_0$$

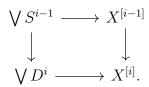
with fibers Y_{ℓ} . Let F = id. Then we obtain a spectral sequences

$$\prod_{\ell} \pi_* Y_{\ell} \Rightarrow \pi_* X.$$

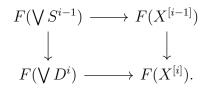
We could apply this to a Postnikov tower for X, but this gives a fairly dumb spectral sequence since we input π_*X to compute π_*X . In general, it's rare to be handed X along with a tower of fibrations as above.

Definition 2.5. We say that F is *linear* or 1-*excisive* if it takes homotopy pushouts to homotopy pullbacks.

Remark 2.6. This definition is motivated by how we build CW complexes. We have homotopy pushouts



If F is linear, then we have a homotopy pullback



The bottom-left is a point (since F is pointed) and the top-left is a product of $F(S^{i-1})$.

The remark implies the following.

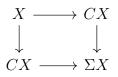
Proposition 2.7. Suppose that F is linear and X is a CW complex. Then there is a spectral sequence

$$\prod_{cells \ of \ X} \pi_* F(S^i) \Rightarrow \pi_* F(X).$$

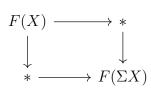
This spectral sequence does not always converge, but with sufficient connectivity and finiteness conditions it does converge.

Example 2.8. We've seen $F = \Omega^{\infty} H\mathbb{Z} \wedge -$ above. We can generalize this by taking $F = \Omega^{\infty}(C \wedge -)$ where C is any spectrum. In fact, these are the only examples! This follows from Brown representability and unwinding the definition of linearity; we will work this out in detail later.

Assume that F is linear. We can apply F to the pushout



to obtain a homotopy pullback



which implies that we have an equivalence

$$F(X) \xrightarrow{\simeq} \Omega F(\Sigma X)$$

Iterating this process, we see that if F is linear, then F(X) is an infinite loop space! This justifies the Ω^{∞} in the example above.

Let \mathcal{L} be the category of linear functors from Top_* to Top_* . To any object of \mathcal{L} , we can attach a spectrum to obtain a functor

$$\mathcal{L} \stackrel{\partial_1}{\to} Sp,$$
$$F \mapsto \partial_1 F$$

where

$$(\partial_1 F)_n = F(S^n)$$

The structure maps were described above, and we see that in fact $\partial_1 F$ is an Ω -spectrum.

We can go the other way by setting

$$Sp \to \mathcal{L},$$

 $C \mapsto \Omega^{\infty}(C \wedge -).$

Theorem 2.9. This gives an equivalence of categories between \mathcal{L} and Sp.

Remark 2.10. We will not give the proof in detail since we don't want to discuss the model structure on \mathcal{L} .

Why should we believe this? If F is linear, then $F \simeq \Omega^{\infty}(\partial_1 F \wedge -)$. Then the spectral sequence

$$\prod_{cells} \pi_* F(S^i) \Rightarrow \pi_* F(X)$$

strongly suggests that the theorem is true.

2.2. Not linear functors? We can now study the composite

$$P_1: Fun(Top_*, Top_*) \xrightarrow{o_1} Sp \to \mathcal{L}$$

and the suspension diagram from before gives a map (after resolving point-set issues)

$$F(X) \to \Omega F(\Sigma X).$$

If F is not linear, this may not be an equivalence. In general, we have

$$P_1F \simeq \Omega^{\infty}(\partial_1 F \wedge -).$$

This is the first polynomial approximation to F. We will discuss the generalization of this next week.

2.3. Connectivity issues.

Definition 2.11. A functor F is $E(c, \kappa)$ if given any diagram

$$\begin{array}{c} X \longrightarrow X_1 \\ \downarrow & \downarrow \\ X_2 \longrightarrow X_{12} \end{array}$$

where $k_i \ge \kappa$ and $X \to X_i$ is k_i -connected, then the map from F(X) to the homotopy limit of

is $(k_1 + k_2 - c)$ -connected.

If F is $E(c, \kappa)$ for some c and κ , we say that F is stably linear.

Example 2.12. If F is linear, then F satisfies $E(-\infty, -1)$.

Example 2.13. If F is the identity, then F satisfies $E(1, \kappa)$ for any κ . This is just a restatement of the Blakers-Massey Theorem.

Definition 2.14. Let $T_1(F)(X)$ be the homotopy limit of

Note that $T_1(F)(X) \simeq \Omega F(\Sigma X)$.

Proposition 2.15. If F is $E(c, \kappa)$, then T, F is $E(c-1, \kappa-1)$.

Proposition 2.16. If F is stably linear, then

$$\operatorname{colim}(F \to T_1(F) \to T_1^{\circ 2}(F) \to \cdots)$$

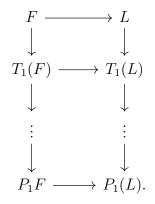
is linear and agrees with P_1F as defined above.

Remark 2.17. The (homotopy) colimit above is defined pointwise on CW complexes. This is one place the finitary condition on F is necessary.

We'll generalize this definition and proposition next week, as well.

Proposition 2.18. The map $F \to P_1F$ is initial (in the homotopy category of functors) amongst linear functors under F. Equivalently, any map from F to a linear functor factors through P_1F .

Proof. Let \mathcal{L} be a linear functor and let $F \to L$ be a map. Then consider



Everything in the right-hand column is equivalent to L, so we're done.

Proposition 2.19. We have

$$P_1(id) \simeq \Omega^{\infty} \Sigma^{\infty}.$$

Proof. We have $T_1(id) \simeq \Omega \Sigma$, and more generally $T_1^{\circ n} \simeq \Omega^n \Sigma^n$. The result follows by letting *n* tend to infinite.

Corollary 2.20. The Hurewicz homomorphism always exists.

Example 2.21. We have $\partial_1(id) \simeq \mathbb{S}$.

Proposition 2.22. Let K be a finite CW complex. Then $P_1Map(K, -) \simeq Map_{Sp}(\Sigma^{\infty}K, \Sigma^{\infty}-).$

In particular,

 $\partial_1 Map(K, -) \simeq DK$

where D is the Spanier-Whitehead dual functor.