GSTS - Orthogonal Calculus

Work with functors

Finite dim'l real vector spaces $\longrightarrow$ pointed spaces

Ex: $F(V) = BO(V)$

Nice: First derivatives generalize Stiefel-Whitney classes
Second derivatives generalize Pontryagin classes

Let $\mathcal{J}$ be category of fin. dim. real vector spaces with inner product.

Study $E: \mathcal{J} \to Top_*$ continuous in the sense that

$$\mathrm{mor}(V,W) \times E(V) \to E(W)$$

is continuous.

Ex: $E(V) = BTop(V)$
$E(V) = BG(S(V))$

Results in orthogonal calculus are usually relative:
doesn't say much about $H^*(BO; \mathbb{Z}/2)$, but does give information about $H^*(-;\mathbb{Z}/2)$ of homotopy fiber of $BO(n) \hookrightarrow BO$.

Derivatives

Def: The first derivative of E is

$$E^{(1)}(V) := \mathrm{hofib}\,[E(V) \xrightarrow{E(\iota)} E(\mathbb{R}\oplus V)]. \quad (\iota: V \longrightarrow \mathbb{R}\oplus V \text{ inclusion})$$

Additional structure:
a binatural transformation

$$\sigma: \underbrace{V^c}_{\text{1-pt compactification}} \wedge E^{(1)}(W) \longrightarrow E^{(1)}(V\oplus W)$$

For $V=0$, W arbitrary, get agreement w/ identity

$$S^0 \wedge E^{(1)}(W) \longrightarrow E^{(1)}(W).$$

Ex: $E(V) = BO(V)$

Then $E^{(1)}(V) \simeq V^c$

and σ is homeomorphism $V^c \wedge W^c \longrightarrow (V \oplus W)^c$

Def: The second derivative is

$$E^{(2)}(V) := \mathrm{hofib}\left[\sigma^{ad} : E^{(1)}(V) \longrightarrow \Omega E^{(1)}(\mathbb{R} \oplus V)\right].$$

Also has additional structure

$$\sigma : (\underbrace{2V}_{\mathbb{R}^2 \otimes V})^c \wedge E^{(2)}(W) \longrightarrow E^{(2)}(V \oplus W)$$

Def: $E^{(3)}(V) := \mathrm{hofib}\left[\sigma^{ad} : E^{(2)}(V) \longrightarrow \Omega^2 E^{(2)}(\mathbb{R} \oplus V)\right]$

etc.

$\rightsquigarrow$ get from E sequence of spectra

$$\Theta E^{(1)}, \dots, \Theta E^{(k)}, \dots$$

where structure maps from suspension of kth term to $(k+1)$st term are specializations of σ.

Can think of $\Theta E^{(i)}$ as the ith derivative of E at ∞.

Ex: $E(V) = BO(V)$

$\Theta E^{(1)} \simeq S^0$

$\Theta E^{(2)} \simeq \Omega S^0$

$\Theta E^{(3)} \simeq \Omega^2 mo(3)$ where $mo(3)$ is $\mathbb{Z}/3$ Moore spectrum.

Ex: $E(V) = BTop(V)$ homeo. $V \to V$

$\Theta E^{(1)}$ is Waldhausen's A-theory of a point

Polynomials

Want: $E^{(n+1)}$ of a functor that is polynomial of degree n to vanish

Prop: For any continuous functor E, $n \geq 0$, $\exists$ natural fibration sequence up to homotopy

$$E^{(n+1)}(V) \xrightarrow{u} E(V) \xrightarrow{p} \underset{0 \neq U \subset \mathbb{R}^{n+1}}{\operatorname{holim}} E(U \oplus V).$$

Def: E is polynomial of degree $\leq n$ if

$$p : E(V) \longrightarrow \underset{0 \neq U \subset \mathbb{R}^{n+1}}{\operatorname{holim}} (U \oplus V)$$

is a homotopy equivalence for all V.

Note:

- codomain of p is the "best guess" for the homotopy type of $E(V)$, knowing only about $E(U \oplus V)$ for $0 \neq U \subseteq \mathbb{R}^{n+1}$

A functor E determines a Taylor tower

$$\begin{tikzcd}
 & & E \arrow[dl, "\eta_{n+1}"'] \arrow[d, "\eta_n"] \arrow[drr, "\eta_0"] & & \\
\cdots \arrow[r] & T_{n+1}E \arrow[r, "r_{n+1}"] & T_n E \arrow[r, "r_n"] & \cdots \arrow[r, "r_1"] & T_0 E
\end{tikzcd}$$

and $r_n \eta_n = \eta_{n-1} : E \longrightarrow T_{n-1}E$.

Thm: For fixed $n \geq 0$, every cts functor E has a universal approx.

$$\eta_n : E \to \underbrace{T_n E}_{\text{polynomial of degree } \leq n}.$$

Often $E : \mathcal{J} \to Top_* \Rightarrow T_n E : \mathcal{J} \to Top_*$

Q: ① For V in $\mathcal{J}$, what is $(T_0 E)(V)$?

② What is hofiber $[r_n : T_n E(V) \longrightarrow T_{n-1}E(V)]$ over base pt?

Answer:

① $(T_0 E)(V) = \underset{j}{\operatorname{hocolim}}\, E(\mathbb{R}^j \oplus V) \simeq \underset{j}{\operatorname{hocolim}}\, E(\mathbb{R}^j) =: E(\mathbb{R}^\infty)$

Thm: The homotopy fiber has the form

$$\Omega^\infty[((nV)^c \wedge \Theta)_{hO(n)}]$$

$$\Theta \simeq \Theta E^{(n)}.$$

Ex: $E(V) = B\underbrace{G(S(V))}_{\substack{\text{grouplike monoid of h.e.} \\ \text{from } S(V) \text{ to itself}}}$

$\eta_1 : BG = E(\mathbb{R}^\infty) \longrightarrow (T_1E)(\mathbb{R}^\infty)$ is h.e.

Also,

$$\begin{array}{ccc} E(0) & \longrightarrow & E(\mathbb{R}^\infty) \\ \downarrow \eta & & \downarrow \eta \\ T_1E(0) & \longrightarrow & T_1E(\mathbb{R}^\infty) \end{array}$$

horizontal map of hofibers by thm has form $G \to \Omega^\infty(\Theta_{hO(1)})$.

$\Theta = \Theta E^{(1)}$ turns out to be sphere spectrum with trivial action of $O(1)$.