HEUTS EXERCISES

TIM CAMPION

Let $F : \mathcal{C} \rightleftarrows \mathcal{C}_n : G$ be an adjunction between pointed, compactly-generated ∞-categories. Recall that this adjunction is a *weak n-excisive approximation* (of $\mathcal{C}$ by $\mathcal{C}_n$) if the following conditions are satisfied:

(1) $\mathrm{id}_{\mathcal{C}_n} : \mathcal{C}_n \to \mathcal{C}_n$ is n-excisive.
(2) $P_n(\mathrm{id}_{\mathcal{C}}) \to GF$ is an equivalence.
(3) $P_n(FG) \to \mathrm{id}_{\mathcal{C}_n}$ is an equivalence.

Exercise 1. *Let $F_{\mathcal{C}} : \mathcal{C} \rightleftarrows \mathcal{C}_n : G_{\mathcal{C}}, F_{\mathcal{D}} : \mathcal{D} \rightleftarrows \mathcal{D}_n : G_{\mathcal{D}}$ be weak n-excisive approximations. Show that there is an equivalence*

$$\begin{gathered} \mathrm{Fun}^{\leq n}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Fun}^{\leq n}(\mathcal{C}_n, \mathcal{D}_n) \\ H \mapsto F_{\mathcal{D}} H G_{\mathcal{C}} \\ G_{\mathcal{D}} K F_{\mathcal{C}} \leftarrow\!\shortmid K \end{gathered}$$

Here $\mathrm{Fun}^{\leq n}$ denotes the category of functors which are n-excisive and commute with filtered colimits.

Hint: Use the fact that $P_n(\Phi\Psi) = P_n(P_n(\Phi)\Psi) = P_n(\Phi P_n(\Psi))$.

Recall that a pointed, compactly-generated ∞-category $\mathcal{C}$ is *n-excisive* if every weak n-excisive approximation of $\mathcal{C}$ is an equivalence. A weak n-excisive approximation of $\mathcal{C}$ by $\mathcal{C}_n$ is a *strong n-excisive approximation* if $\mathcal{C}_n$ is n-excisive.

Exercise 2. *(1) Let $\mathcal{C}$ be a pointed compactly-generated ∞-category. Show that the weak n-excisive approximations of $\mathcal{C}$ form a (possibly large) preorder, and that a strong n-excisive approximation of $\mathcal{C}$ is a maximal element of this preorder.*
(2) Let $n < 0$ be an integer. Show that the category of n-connective spectra is a weak 1-excisive approximation to Top_*, but not a strong 1-excisive approximation. Show that the category of spectra is a strong 1-excisive approximation to Top_*.

In my talk, I said some things about the functoriality of $\mathbb{P}_n$, not all of which were correct.

Exercise 3. *Let $\mathcal{C}, \mathcal{D}$ be pointed, compactly-generated ∞-categories.*
(1) If $F : \mathcal{C} \to \mathcal{D}$ is a functor preserving filtered colimits, define $\mathbb{P}_n F = \Sigma^\infty_n P_n F \Omega^\infty_n$. Show that $\mathbb{P}_n F = P_n(\mathsf{Lan}_{\Sigma^\infty_n} \Sigma^\infty_n F)$. In my talk, I claimed that $\mathbb{P}_n F = \mathsf{Lan}_{\Sigma^\infty_n F} \Sigma^\infty_n F$. What's wrong with this claim?

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(2) *Note that $\mathbb{T}_n$, and hence $\mathbb{P}_n$, is functorial in left adjoint functors. Show that when F is a left adjoint, the resulting definition of $\mathbb{P}_n F$ agrees with the definition of the first part.*