THH IS THE FIRST DERIVATIVE OF K-THEORY

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1. K-theory

Today we'll be talking about the directional derivative of K-theory. We begin with a few definitions.

Recall that *topological K-theory* is concerned with complex vector bundles, i.e. families of $\mathbb{C}^n$ parametrized by a space X. A rank n complex vector bundle $V \to X$ is classified by a map $X \to BU(n)$, where $BU(n)$ may be thought of as a delooping of the category of vector spaces of dimension n. In other words, families of vector spaces parametrized by X are classified by maps

$$X \to \sqcup_n BU(n).$$

The K-theory of $\mathbb{C}$, $K(\mathbb{C})$, is designed to classify virtual vector bundles over X. It arises as the group completion of $\sqcup_n BU(n)$. This group completion may be identified with $\Omega B(\sqcup_n BU(n))$. A calculation shows that $\Omega B(\sqcup_n BU(n))$ is equivalent to $\mathbb{Z} \times BU$.

More generally, the K-theory $K(R)$ of a ring R classifies "virtual R-bundles over a space X". More precisely, we have

$$K(R) \simeq \Omega BProj(R) \simeq K_0(R) \times BGL_\infty(R)^+$$

where $Proj(R)$ is the category of finite rank projective R-modules and $(-)^+$ is Quillen's plus construction.

We now ask what $\Omega B\mathcal{C}$ is, where $\mathcal{C} = (Proj(R), \oplus)$ or $\mathcal{C} = (Vect(\mathbb{C}), \oplus)$, or more generally $\mathcal{C}$ is a symmetric monoidal groupoid. First, the category $B\mathcal{C}$ has a single object $*$ and 1-morphisms $\oplus M$, $M \in ob(\mathcal{C})$, and 2-morphisms $Mor(\mathcal{C})$. So, we are really interested in $\Omega|B\mathcal{C}|$.

Now, let's review the categorical nerve $NB\mathcal{C}$. The n-th level $N_n B\mathcal{C}$ consists of sequences of n objects $M_1, \ldots, M_n \in ob(\mathcal{C})$. The face and degeneracy maps are the same as those for a group or monoid.

Definition 1.1. Given a symmetric monoidal category $\mathcal{C}$, we define $S_\bullet \mathcal{C}$ to be the simplicial symmetric monoidal category where $S_n\mathcal{C}$ consists of sequences of n objects $M_1, \ldots, M_n \in \mathcal{C}$ in the form

$$M_1 \to M_1 \otimes M_2 \to M_1 \otimes M_2 \otimes M_3 \to \cdots \to M_1 \otimes \cdots \otimes M_n.$$

Face and degeneracy maps are as above.

Remark 1.2. *There is an equivalence of simplicial categories $N_\bullet B\mathcal{C} \simeq S_\bullet \mathcal{C}$.*

Example 1.3. Let M be a monoid. Then there is a map $M \to \Omega BM$. More generally, if $\mathcal{C}$ is a topological category, then there is a map of spaces $Mor(\mathcal{C}) \to P|\mathcal{C}|$ where P is the path space functor.

There is a map from the 1-skeleton of $S_\bullet\mathcal{C}$ to all of $S_\bullet\mathcal{C}$. But note that the 1-skeleton is just $(\Delta^1 \times S_1\mathcal{C})/(d_0, d_1)$, which is equivalent to $S^1 \times S_1\mathcal{C}$. By adjunction, we then have a map

$$|\mathcal{C}| = S_1\mathcal{C} \to \Omega S_\bullet \mathcal{C}.$$

This map is a group completion.

Definition 1.4. We define the *algebraic K-theory* $K(\mathcal{C})$ of a symmetric monoidal category $\mathcal{C}$ is defined by

$$K(\mathcal{C}) := \Omega|S_\bullet\mathcal{C}|.$$

2. Stable K-theory and THH

Since functor calculus is interested in functors from spaces, we need to understand K-theory as a functor from spaces.

Let A be an abelian group. Let $S^n_\bullet$ denote the simplicial set model for the n-sphere S^n. Let $A[S^n_\bullet]$ be a simplicial abelian group with $|A[S^n_\bullet]| \simeq K(A, n)$ (compare with Dold-Kan).

Definition 2.1. Let A be a ring and let V be a simplicial module. Let $A \rtimes V$ be the ring with $(a, v) \cdot (b, w) = (ab, aw + bv)$. We then define

$$K(A, V) := |K(A \rtimes V)|.$$

We define *stable K-theory*

$$K^s(A, V)$$

to be the "derivative of K at A in the direction M". More precisely, we have

$$K^s(A, M) := fib(\operatorname{colim}_n \Omega^{n+1} K(A \rtimes M[S^n_\bullet]) \to K(A)).$$

Definition 2.2. We define $THH(A)$ to be the geometric realization of the cyclic bar construction $B^{cyc}_\bullet(A)$.

One also has

$$THH(A) \simeq |N^{cyc}_\bullet Proj(A)|.$$

This motivates the definition:

Definition 2.3. We define $THH(\mathcal{C}) = |N^{cyc}_\bullet \mathcal{C}|$.

One also has

$$THH(Proj(A)) \simeq \Omega THH(S_\bullet Proj(A)).$$

One can define the *Dennis trace*

$$K(\mathcal{C}) \to THH(\mathcal{C})$$

using these equivalences and the map

$$\Omega|S_\bullet\mathcal{C}| \to \Omega|THH S_\bullet \mathcal{C}|.$$

One can then show that the directional derivatives of both sides coincide.