GOODWILLIE CALCULUS OF RATIONAL SPACES

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Kierkegaard: "People understand me so poorly that they don't even understand my complaint aobut them not understanding me."

1. Introduction

Recall that in order to study topological spaces, we can instead study simplicial sets since

$$Top_* \simeq sSet_*.$$

We want to do this (sometimes) since we want to use combinatorial arguments, and this is sometimes more natural in simplicial sets.

A step up from combinatorial arguments is algebraic arguments, since then we have an additional structure to work with.

Example 1.1. We can think of singular cochains $C^*(X;\mathbb{Z})$ as a model for X. However, it's very hard to extract $\pi_1(X)$ from this since it (usually) only sees abelian group data. There is also a multiplicative structure encoded by cup product, but this is a lot of data to keep track of.

In order to resolve the first issue, we work with 1-connected spaces. However, there is no way of strictifying the singular cochains into an actual commutative algebra. To resolve this, we throw away more data.

Definition 1.2. If $f : X \to Y$ is a map with X, Y 1-connected, then the following are equivalent:

(1) $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism.
(2) $H_*(f;\mathbb{Q})$ is an isomorphism.
(3) $H^*(f;\mathbb{Q})$ is an isomorphism.

If so, we say that f is a *rational equivalence*.

Now, we want to think of $C^*(X;\mathbb{Q})$ as a model for X as a commutative algebra. However, we're still not quite there. The de Rham complex gives a good model, but we don't want to restrict X to being a smooth manifold.

Definition 1.3. We define $\mathfrak{a}^*_\bullet$ to be a simplicial differential graded commutative algebra (sdgca) by setting

$$\mathfrak{a}^*_n = (\mathbb{Q}[t_0, \dots, t_n, dt_0, \dots, dt_n/J_n, d)$$

where $|t_i|0$, $d(t_i) = dt_i$, and $J_n = -(1 - \sum t_i, \sum dt_i)$. The simplicial structure maps are given by

$$d_i(t_k) := \begin{cases} t_k & k < i, \\ 0 & k = i, \\ t_{k-1} & k > i, \end{cases}$$

and

$$s_i(t_k) := \begin{cases} t_k & k < i, \\ t_k + t_{k+1} & k = i, \\ t_{k+1} & k > i. \end{cases}$$

Fact 1.4. *We have*

$$\Omega^*_{dR}(\Delta^n) \cong C^\infty(\Delta^n) \otimes_{\mathfrak{a}_n^0} \mathfrak{a}_n^*.$$

This suggests an algebraic model for the rational homotopy type of a space X.

Definition 1.5. We define

$$\mathcal{A}^* : sSet \to dgCA$$

by setting

$$\mathcal{A}^*(K) := Hom_{sSet}(K, \mathfrak{a}^*_\bullet).$$

Fact 1.6. *We have*

$$H^*(K; \mathbb{Q}) \cong H^*(\mathcal{A}^*(K))$$

as graded commutative algebras.

This can be proven using that we know it for Δ^n. One can then prove it for spheres, and then prove it for general K by CW approximation.

Theorem 1.7 (Sullivan)**.** *The map*

$$sSet^{\geq 1}_{\mathbb{Q}} \xrightarrow{\mathcal{A}^*} dgCA^{\geq 1}_{\mathbb{Q}}$$

is a Quillen equivalence.

Fact 1.8. *Consider the diagram*

$$\begin{array}{ccc} Sp & \xrightarrow{C^*_{CW}(-;\mathbb{Q})} & Ch_{\mathbb{Q}} \\ {\scriptstyle \Sigma^\infty}\big\updownarrow{\scriptstyle \Omega^\infty} & & {\scriptstyle Forget}\big\updownarrow{\scriptstyle Free} \\ Top_* & \xrightarrow{\mathcal{A}^*} & dgCA. \end{array}$$

Here, $N^*(-;\mathbb{Q})$ is the normalized chain complex defined by taking the diagonal of the normalized chains of each i-cell of a CW spectrum. This diagram definitely doesn't commute, but it is "something like homotopy commutative."

Fact 1.9. *Consider the Samelson bracket*

$$\Omega X \wedge \Omega X \xrightarrow{[-,-]} \Omega X,$$
$$(\alpha, \beta) \mapsto \alpha\beta\alpha^{-1}\beta^{-1}.$$

Note that Moore loops can be used to make sense of associativity of this operation. This makes $\pi_*\Omega X$ a graded Lie algebra.

Theorem 1.10 (Quillen)**.** *There exists a functor*

$$\omega : sSet_{*,\mathbb{Q}}^{>1} \to dgLie^{>0}$$

which is a Quillen equivalence. Moreover,

$$\pi_*\Omega X \simeq H_*(\omega X).$$

Moreover, we can consider the diagram

$$\begin{tikzcd} Sp \arrow[r, "C_*^{CW}(-;\mathbb{Q})"] \arrow[d, leftrightarrow] & Ch_{\mathbb{Q}} \arrow[d, shift left, "Triv"] \\ Top_* \arrow[r, "\omega"] & dgLie. \arrow[u, shift left, "(-)^{ab}"] \end{tikzcd}$$

This is again "something like homotopy commutative."

Theorem 1.11 (Pereira)**.** *If $\mathcal{O}$ is a (reduced) operad in Sp or Ch, then*

$$P_n(id_{Alg_{\mathcal{O}}})(A) \simeq \mathcal{O}_{\leq n} \circ_{\mathcal{O}} A.$$

Further,

$$\partial_*(Id_{Alg_{\mathcal{O}}}) \simeq \mathcal{O}$$

as a symmetric sequence.

Roughly speaking, the right-hand side of the first equation in the theorem says to kill off everything in A that looks like it's of arity higher than n.

Conjecture 1.12. *The second statement of the theorem is true as operads.*

Definition 1.13. If $L \in dgLie$, define $\Gamma^2(L) := [L, L]$ and

$$\Gamma^{n+1}(L) := [L, \Gamma^n(L)].$$

Define

$$\tilde{P}_n(L) := L/\Gamma^{n+1}(L).$$

Theorem 1.14 (B. Walter)**.** *There is an equivalence*

$$P_n(id_{dgLie^{>0}}) \simeq \tilde{P}_n.$$

Thus, there is an equivalence

$$\partial_*(id_{dgLie^{>0}}) \simeq Lie$$

as operads.

Proof. If $L \in dgLie^{>0}$, then there exists $V \in grVect^{>0}$ such that

$$L \simeq (Lie(V), d)$$

(we do not need to know the definition of d today).

We first show that $\tilde{P}_n$ is n-excisive. To show this, we check that

$$F_n : fib(\tilde{P}_n(Lie(V)) \to \tilde{P}_{n-1}(Lie(V)))$$

is n-homogeneous. First, note that $F_n(Lie(V))$ consists of Lie-words of length precisely n. This is the same thing as $(Lie(n) \otimes V^{\otimes n})_{\Sigma_n}$ (note that this is the minimal model). It follows that this is n-homogeneous by tracing through the definitions.

Now

$$(Lie(V), d) \to \tilde{P}_n(Lie(V), d)$$

is a $(n+1) \cdot (conn(V))$-connected, i.e. they agree up to degree n. This means that the two functors agree up to order n, but since the right-hand side was n-excisive, this implies the desired equivalence. □

Remark 1.15. *This result would be implied by Pereira's theorem above plus the conjecture by taking $\mathcal{O} = Lie$.*

Theorem 1.16 (Pereira)**.** *If*

$$F : \mathcal{C} \leftrightarrows \mathcal{D} : G$$

is a Quillen equivalence, then

$$P_n(id_{\mathcal{C}}) \simeq G \circ P_n(id_{\mathcal{D}}) \circ F$$

and vice versa.

Theorem 1.17 (Quillen-Sullivan)**.** *Let X be a 1-connected finite complex. Then*

$$\pi_*(X) \otimes \mathbb{Q} \cong AQ^*(\Lambda_X)$$

where AQ^* is André-Quillen homology and Λ_X is a minimal model for X. Moreover,

$$AQ^*(\Lambda_X) \simeq H^*(sLie(\Lambda_X), d_{Har})$$

where $d_{Har}(xy) = [x, y]$ is the Harrison differential.

This looks like the Goodwillie spectral sequence! More precisely, we would like to say that $sLie(\Lambda_X)$ is the E_1-page of the Goodwillie spectral sequence and the Harrison differential is the d_1-differential. The spectral sequence would then collapse at E_2 to give the equivalence. It may be the case that the homology should be moved inside, but this will be discussed after the talk...

Finally, we should ask what happens when we try to remove 'rational' from the assumptions. This would lead to Mandell's work on p-adic homotopy theory, the work of Behrens and Rezk on convergence of the $K(n)$-local Goodwillie spectral sequence, and so on.