EXERCISES FOR WEEK 14

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(1) Determine the types of $\mathbb{S}$, $\Sigma^\infty \mathbb{RP}^2$, and $\Sigma^\infty \mathbb{RP}^2 \wedge \mathbb{CP}^2$.

(2) (The Bousfield-Kuhn functor) Fix $n > 0$. Let (M, v) be a space of type n and set
$$v^{-1}\pi_*(X; M) := \mathrm{colim}_r [\Sigma^{rd} M, X]_*.$$
Define a functor $\Phi_v : Top \to Sp$ as follows. If $n \equiv -e \mod d$, with $0 \leq e \leq d-1$, then set
$$\Phi_v(Z)_n := \Omega^e Map_{Top}(M, Z).$$
The structure maps are the identity unless $n \equiv 0 \mod d$, in which case it is given by
$$v(Z) : Map_{Top}(M, Z) \xrightarrow{v^*} Map_{Top}(\Sigma^d M, Z) \simeq \Omega^d Map_{Top}(M, Z).$$
Prove the following:
   (a) $\pi_*(\Phi_v(Z)) \cong v^{-1}\pi_*(Z; M)$.
   (b) If a map $Y \to Z$ induces an isomorphism on π_* for $* >> 0$, then $\Phi_v(Y) \to \Phi_v(Z)$ is a stable equivalence. In particular, the r-connected covering map $Z\langle r \rangle \to Z$ induces a stable equivalence $\Phi_v(Z\langle r \rangle) \to \Phi_v(Z)$ for all r.
   (c) $v^* : \Phi_v(Z) \to \Phi_{\Sigma^d v}(Z)$ is a stable equivalence.
   (d) If $v_0, v_1 : \Sigma^d M \to M$ are homotopic maps, then $\Phi_{z_0}(Z)$ is naturally stably equivalent to $\Phi_{v_1}(Z)$.
   (e) Using the asymptotic uniqueness of v_n-self-maps, show that the Φ_v is independent of the choice of v_n-self-map.

(3) Let $T(n) := v^{-1}M$ with (M, v) as above. Define $T(\leq n) := T(0) \vee \cdots \vee T(n)$. Let $L_n^f := L_{T(\leq n)}$.
   (a) Show that L_n^f may be identified as a Bousfield localization.
   (b) Show that L_n^f is smashing, i.e. $L_n f X \simeq X \wedge L_n^f S^0$.
   (c) Show that Bousfield localization $L_E : Sp \to Sp$ is finitary if and only if L_E is smashing.

(4) Let $C_n^f A := fib(X \to L_n^f A)$. The *$v$-periodic homotopy groups of a space X* may be defined by
$$v^{-1}\pi_*(X; M) := \pi_*(C_{n-1}^f \Phi_v(X)).$$

(5) Prove that there exists a sequence of type n finite spectra
$$F(1) \to F(2) \to \cdots$$
such that the map
$$S^0 \to F(\infty)$$

induces an isomorphism in $T(m)_*$-homology for all $m \geq n$. The *Telescopic Bousfield-Kuhn functor* $\Phi_n^T : Ho(Top) \to Ho(Sp)$ is defined by the formula

$$\Phi_n^T(Z) := \underset{\longleftarrow}{\text{holim}}_k \, \Phi(F(k), Z).$$

(6) Show that $\Phi_n^T(\Omega^\infty X) \simeq L_{T(n)} X$ for all $X \in Ho(Top)$.
(7) Give a definition of v_n-periodic homotopy groups of spaces which does not depend on a choice of finite complex.
(8) Recall the tower of fibrations which filtered $\mathbb{D}_n(X) \simeq Map_*(SK_n, \Sigma^\infty X^\wedge n)_{h\Sigma_n}$. Use this to show that if X is an odd-dimensional sphere, then $\mathbb{D}_n(X)$ is rationally trivial for $n > 1$.
(9) Let $A[k-1]$ be the subalgebra of the Steenrod algebra generated by the first k Milnor primitives $P_1^0, \ldots, P_1^r$ or (if $p > 2$) $Q_0, \ldots, Q_k$. Show that if $H^*(X)$ is free over $A[k-1]$, then $v_i^{-1}\pi_*(X) = 0$ for any $i \leq k-1$.