MINICOURSE PART II

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1. Higher analogs of linear functors

Last week we talked about linear functors. Our goal today is to generalize to higher cubes. Unfortunately, today will mostly be definitions, but hopefully they'll make sense given what we did last week.

Definition 1.1. An *n-cube* is a functor

$$C : P(\underline{n}) \to \mathcal{C}$$

where $\underline{n} = \{1, \ldots, n\}$. Such a functor is *cartesian* if the map

$$C(\emptyset) \to \lim_{P(\underline{n})\setminus\{\emptyset\}} C$$

is an equivalence.

Example 1.2. If a 2-cube is cartesian, then it is a pullback square.

Definition 1.3. A n-cube is *cocartesian* if

$$\operatorname{colim}_{P(\underline{n})-\underline{n}} C \to C(\underline{n})$$

is an equivalence. It is *strongly cocartesian* if it is cocartesian on each face of dimension at least two.

Example 1.4. (a picture of a 3-cube)

Definition 1.5. A functor $F : \mathcal{C} \to \mathcal{D}$ is *n-excisive* if it takes strongly cocartesian $(n+1)$-cubes to cartesian cubes.

Exercise 1.6. *The functor $\Omega^\infty(C \wedge (-)^{\wedge n}))$ (from spaces to spaces) is n-excisive for any spectrum C.*

Last week we defined linear approximations. The key idea was that if we linearized with respect to the diagram

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X, \end{array}$$

then we linearized with respect to everything. Our goal now is to define the higher-dimensional analog of this diagram

Definition 1.7. Let $X, Y \in Top_*$. The *join* of X and Y is

$$X * Y = (X \times Y \times I)/\sim$$

where $(x_0, y, 0) \sim (x_1, y, 0)$ and $(x, y_0, 1) \sim (x, y_1, 1)$.

Example 1.8. We have:

(1) $X * \emptyset \simeq X$,
(2) $X * \{pt\} \simeq CX$,
(3) $X * \{pt\} \sqcup \{pt\} \simeq \Sigma X$,
(4) $X * \{pt\} \sqcup \{pt\} \sqcup \{pt\} \simeq$ (ΣX with an extra pointy bit).

Now let X be a space. We define

$$C^n_X : P(\underline{n}) \to Top,$$
$$u \mapsto X * U.$$

Note that when $n = 2$, we recover precisely the preferred diagram from above.

Remark 1.9. *The join commutes with pushouts, so the functor above is strongly cocartesian.*

Let's use this generalization to define higher analogs of the T-functors from the linear case. Given a homotopy functor F, we want to consider

$$F(X) \simeq F(X * \emptyset) \to \lim_{P(\underline{n+1})-\emptyset} F \circ C^{n+1}_X.$$

We define

$$T_n F := \lim_{P(\underline{n+1})-\emptyset} F(C^{n+1}_X)$$

to be the left-hand side.

Remark 1.10. *If F is n-excisive, then this map is an equivalence. This follows from tracing through the definitions, along with the remark above that C^{n+1}_X is strongly cocartesian.*

Definition 1.11. A functor F satisfies $E_n(c, \kappa)$ if for any strongly cocartesian cube $C(\underline{n+1}) \to Top_*$ with $C(\emptyset) \to C(\{i\})$ is k_i-connected with $k_i \geq \kappa$, then

$$F(C(\emptyset)) \to \lim_{P(\underline{n+1})-\emptyset} F \circ C$$

is $(\sum_i k_i - c)$-connected.

We say that F is *stably n-excisive* if it is $E_n(c, \kappa)$.

Remark 1.12. *F is n-excisive if and only if F is $E_n(-\infty, -\infty)$.*

Definition 1.13. We say that $F \to G$ is $\mathcal{O}_n(c, \kappa)$ if for any k-connected X with $k \geq \kappa$, then $F(X) \to G(X)$ is $((n+1)k - c)$-connected.

If F and G satisfy $\mathcal{O}_n(c, \kappa)$, then we say that they *agree up to n-th order*.

Proposition 1.14. *If F is stably excisive, say F is $E_n(c, \kappa)$, then:*

(1) *$T_n(F)$ is $E_n(c-1, \kappa-1)$.*
(2) *$F \to T_n(F)$ is $\mathcal{O}(c, \kappa)$.*
(3) *$P_n F := \operatorname{colim}(F \to T_n(F) \to T_n(T_n(F)) \to \cdots)$ is n-excisive.*

(4) *$F \to P_nF$ agrees up to n-th order.*

The proof of this proposition is very similar to the proof of the linear analog from the problem session. The key point is that Ω commutes with homotopy limits.

Proposition 1.15. *The functor P_n commutes with finite homotopy limits and filtered homotopy colimits.*

This follows from running through the definitions and seeing when homotopy limits and colimits commutes.

Theorem 1.16. *In the homotopy category of functors, the map $F \to P_nF$ is initial amongst maps to n-excisive functors.*

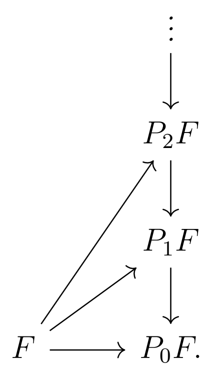
The proof is, again, similar to the linear case which was proven last week.

2. The Taylor tower

Lemma 2.1. *If F is n-excisive, then it is also $(n+1)$-excisive.*

Hint of proof. Think of a strongly cocartesian $(n+2)$-cube as being built out of several strongly cocartesian $(n+1)$-cubes, then apply F and figure out what commutes. □

Corollary 2.2. *We have a tower*



Note that $P_0F \simeq F(pt)$. This is called the Taylor tower*.*

Theorem 2.3. *If F is $E_n(n\rho - q, \rho + 1)$ for some fixed ρ and q and for all n, we call F ρ-*analytic*. If X is k-connected with $k > \rho$, then*

$$F(X) \to \lim P_n(F)(X)$$

is an equivalence. Moreover, the spectral sequence associated to the Taylor tower converges strongly.

The statement about strong convergence comes from the connectivity assumptions above. In other words, everything was set up to give this nice property!

Definition 2.4. Let

$$D_n(F) := hofib(P_nF \to P_{n-1}F).$$

A functor F is *n-homogeneous* if it is n-excisive and if $P_{n-1}F \simeq *$.

Lemma 2.5. *D_nF is n-homogeneous and D_1F is linear.*

Proof. This follows from the equivalence $P_{n-1}(P_nF) \simeq P_{n-1}F$. □

We saw last week that all linear functors were classified by spectra. Our goal now is to get the analogous result for n-homogeneous functors.

Remark 2.6. *For any functor F, we can construct BD_nF such that*

$$P_nF \to P_{n-1}F \to BD_nF$$

is a fiber sequence. This implies that

$$D_nF \simeq \Omega BD_nF.$$

In fact, there is a general method for delooping n-homogeneous functors, but we will not cover this today.

Theorem 2.7. *Let $\mathcal{H}_n(\mathcal{C}, \mathcal{D})$ be the category of n-homogeneous functors. Then*

$$\mathcal{H}_n(Top_*, Sp) \xrightarrow{\Omega^\infty} \mathcal{H}_n(Top_*, Top_*)$$

is an equivalence.

In other words, every n-homogeneous functor is an infinite loop space.

Lemma 2.8. *Suppose that $\mathcal{C}^n \xrightarrow{F} \mathcal{D}$ is linear in each variable, and suppose also that F is* symmetric, *i.e. for each $\sigma \in \Sigma_n$, there is an equivalence $\sigma_* : F(X_1, \ldots, X_n) \to F(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$. Then we can define*

$$\Delta_n F(X) := F(X, \ldots, X)_{h\Sigma_n}$$

and this is n-homogeneous.

Let $\mathcal{L}_n(\mathcal{C}, \mathcal{D})$ be the category of functors which are linear in each variable and symmetric (as above). We have

$$\mathcal{L}_n(\mathcal{C}, \mathcal{D}) \xrightarrow{\Delta_n} \mathcal{H}_n(\mathcal{C}, \mathcal{D}).$$

We want to define the cross effect as the inverse of this.

Remark 2.9. *We now recall the classical notion of cross effect. Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then the second cross effect of f is*

$$cr_2(f)(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2) + f(0).$$

Suppose that $f(0) = 0$ to simpify. Then $cr_2(f)$ vanishes precisely when f is linear.

More generally, the n-th cross effect is used to check if f is a polynomial of degree n.

We want to construct a functor cr_n which