MINICOURSE PART III

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1. Cross-effects

Last time, we discussed a map

$$\mathcal{L}_n(\mathcal{C}, \mathcal{D}) \overset{\Delta_n}{\to} \mathcal{H}_n(\mathcal{C}, \mathcal{D})$$

where the left-hand side is the category of functors $F : \mathcal{C}^n \to \mathcal{D}$ which are symmetric and linear in each variable, and the right-hand side is the category of n-homogeneous functors. we think of the left-hand side as multilinear functors, whereas the right-hand side are homogeneous functors. Recall that

$$\Delta_n F(X) = F(X, \dots, X)_{h\Sigma_n}.$$

Recall from last time that we defined the cross-effect in calculus by

$$cr_2(f)(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2) + f(0).$$

In this setting, we should replace the symbol $+$ by $\vee$ and replace $-$ by $fib(-)$.

Definition 1.1. Let $X_i \in \mathcal{C}$. Define an n-cube

$$C_n(X_1, \dots, X_n) : P(\underline{n}) \to \mathcal{C}$$

by

$$U \mapsto \bigvee_{i \in \underline{n} - U} X_i.$$

Example 1.2. We have

$$C_1(X) : X \mapsto *$$

and $C_2(X_1, X_2)$ is given by

$$\begin{array}{ccc} X_1 \vee X_2 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & *. \end{array}$$

Definition 1.3. Let $F : \mathcal{C} \to \mathcal{D}$. Then $cr_n(F) : \mathcal{C}^n \to \mathcal{D}$ is defined by

$$cr_n(F)(X_1, \dots, X_n) := tfib(F \circ C_n(X_1, \dots, X_n))$$

where $tfib$ is the *total fiber*.

Example 1.4. We have

$$cr_1(F)(X) = fib(F(X) - F(pt)).$$

Example 1.5. We want to understand $cr_2(F)(X)$. Let

$$F_2 = fib(F(X_1 \vee X_2) \to F(X_1)),$$

$$F_1 = fib(F(X_2) \to F(pt)).$$

Then

$$cr_2(F)(X) = fib(F_2 \to F_1).$$

We are almost ready to define the inverse to Δ_n. First, we need the following observation:

Remark 1.6. *The functor $cr_n(F)$ is a symmetric functor from $\mathcal{C}^n \to \mathcal{D}$.*

In particular, we have

$$cr_n : Fun(\mathcal{C}, \mathcal{D}) \to Fun^{\Sigma_n}(\mathcal{C}^n, \mathcal{D}).$$

Proposition 1.7. *If F is n-excisive, then $cr_m(F)$ is $(n-m+1)$-excisive in each variable.*

In particular, this implies that we have a functor

$$cr_n : \mathcal{H}_n(\mathcal{C}, \mathcal{D}) \to \mathcal{L}_n(\mathcal{C}, \mathcal{D}).$$

Theorem 1.8. *This gives an equivalence of categories.*

2. Derivatives

Recall from the first lecture that

$$\mathcal{L}_1(\mathcal{C}, \mathcal{D}) \simeq Sp$$

where $\mathcal{C}$ and $\mathcal{D}$ were spaces or spectra. This equivalence was defined by

$$F \mapsto \{F(S^n)\},$$

with the structure maps $F(X) \to \Omega F(\Sigma X)$ equivalences by linearity. The inverse was defined by

$$C \mapsto \Omega^\infty C \wedge -.$$

Remark 2.1. *Under some mild hypotheses on $\mathcal{C}$ and $\mathcal{D}$, the above equivalence still holds if we replace Sp by the stabilization of $\mathcal{D}$.*

Our goal now is to mimic this for multilinear functors.

Fact 2.2. *If $F \in \mathcal{L}_n(\mathcal{C}, \mathcal{D})$, then the map*

$$F(X_1, \dots, X_n) \to \Omega^n F(\Sigma X, \dots, \Sigma X)$$

is an equivalence. We may rewrite the right-hand side as $\Omega^{1+\cdots+1} F(\Sigma X, \dots, \Sigma X)$ more suggestively, and it turns out that this map is a Borel equivalence.

This fact implies that we have a functor

$$\mathcal{L}_n(\mathcal{C}, \mathcal{D}) \to Sp^{h\Sigma_n}$$

defined by

$$F \mapsto \{F(S^k, \ldots, S^k)\}.$$

The structure maps

$$F(S^k, \ldots, S^k) \to \Omega^{1+\cdots+1} F(S^{k+1}, \ldots, S^{k+1})$$

are Borel equivalences, and in fact we can identify $\Omega^{1+\cdots+1}$ with Ω^ρ where ρ is the regular representation of Σ_n. Above, $Sp^{h\Sigma_n}$ is the category of Borel Σ_n-spectra.

We can define an inverse

$$Sp^{h\Sigma_n} \to \mathcal{L}_n(Top_*, Top_*)$$

by sending

$$\mathcal{C} \mapsto \big((x_1, \ldots, x_n) \mapsto \Omega^\infty_{\Sigma_n} C \wedge X_1 \wedge \cdots \wedge X_n\big).$$

This turns out to define an equivalence of categories.

Corollary 2.3. *We have*

$$\partial_n : \mathcal{H}_n(\mathcal{C}, \mathcal{D}) \xrightarrow{\simeq} Sp^{h\Sigma_n}$$

with inverse

$$C \mapsto (X \mapsto \Omega^\infty (C \wedge X^{\wedge n})_{h\Sigma_n}).$$

Recall that $D_n F(X) = hofib(P_n F(X) \to P_{n-1} F(X))$. Then

$$D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

for some $\partial_n F \in Sp^{h\Sigma_n}$.

Corollary 2.4. *If F is ρ-analytical and X is k-connected for $k > \rho$, then there is a strongly convergent spectral sequence (see first lecture) of the form*

$$\prod_n \pi_* D_n F(X) \cong \prod_n \pi^s_*((\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}) \Rightarrow \pi_* F(X).$$

Why is this cool? The left-hand side is an object in *stable* homotopy theory! We then have tools like the Serre spectral sequence and the homotopy orbit spectral sequence to analyze this! Of course, this means that the derivatives in the Goodwillie spectral sequence must be hard, since they have to encode *unstable* information.

3. Computations, examples, and an overview of recent research

For now, assume that all functors are reduced.

Theorem 3.1. *There is an equivalence*

$$cr_n P_n F \simeq cr_n D_n F.$$

Fact 3.2. *We have*

$$Fun^{\Sigma_n}(\mathcal{C}^n, \mathcal{D}) \xrightarrow{P_{1,\ldots,1}} \mathcal{L}_n(\mathcal{C}, \mathcal{D}).$$

In particular, we have a commutative diagram (up to homotopy

$$\begin{array}{ccccc} Fun(\mathcal{C},\mathcal{D}) & \xrightarrow{cr_n} & Fun(\mathcal{C}^n,\mathcal{D}) & \xrightarrow{P_{1,\ldots,1}} & \mathcal{L}_n(\mathcal{C},\mathcal{D}) \\ \downarrow{\scriptstyle D_nF} & & & & \downarrow{\scriptstyle \simeq} \\ \mathcal{H}_n(\mathcal{C},\mathcal{D}) & & \xrightarrow{\qquad\partial_n,\simeq\qquad} & & Sp^{\Sigma_n}. \end{array}$$

Example 3.3. Let's compute $\partial_2(id)$. First, we have

$$cr_2(id)(X_1, X_2) \simeq fib(X_1 \vee X_2 \to X_1 \times X_2).$$

If we took the cofiber instead, then we'd have $X_1 \wedge X_2$.

Now we need to stabilize. In a stable category (e.g. spectra), cofiber sequences are fiber sequences. Therefore we can show that

$$P_{1,1}cr_2(id)(X_1, X_2) \simeq \Omega^\infty(\Omega(\Sigma^\infty X_1 \wedge \Sigma^\infty X_2)).$$

Since $\Omega(-) \simeq S^{-1} \wedge -$, we have

$$\partial_2(id) \simeq S^{-1}.$$

We can then go further:

$$D_2(id)(X) \simeq \Omega^\infty(S^{-1} \wedge X^{\wedge 2}_{h\Sigma_2}).$$

For now, let's omit id from the notation. Then we have a fiber sequence

$$D_2(X) \to P_2(X) \to P_1(X) \to BD_2(X)$$

where we have used the fact that $D_2(X)$ is an infinite loop space in order to deloop. We can then identify the last map above with the *James-Hopf map*

$$\Omega^\infty\Sigma^\infty X \to \Omega^\infty(\Sigma^\infty X^{\wedge 2}_{h\Sigma_2}).$$

This implies that $\pi_* P_2(X)$ is the *metastable homotopy groups of X*, which were studied by Whitehead, Toda, Mahowald, etc...

The higher levels of the tower don't have names, but we can still ask about them.

Theorem 3.4 (Johnson)**.** *There is an equivalence (not Σ_n-equivariant)*

$$\partial_n(id) \simeq \bigvee_{(n-1)!} S^{1-n}.$$

Theorem 3.5 (Arone-Mahowald)**.**
(1) *If m is odd, then*

$$\partial_n \wedge_{h\Sigma_n} (S^m)^{\wedge n} \simeq *$$

if $n \neq p^d$ for any prime p.

(2) *We have*

$$\partial_{p^d} \wedge_{h\Sigma_{p^d}} (S^m)^{\wedge p^d}$$

are p-local.

(3) *The mod p cohomology $H^*(\partial_{p^d} \wedge_{h\Sigma_{p^d}} (S^m)^{p^d}; \mathbb{F}_p)$ is "computable and nice."*

Remark 3.6. *A monograph of Behrens recovers Toda's unstable computations using the Goodwillie spectral sequence and the previous theorem.*

Theorem 3.7 (Arone-Ching)**.** *The derivatives of the identity functor ∂_* has the structure of an* operad*, i.e. maps*

$$\partial_n \wedge \partial_{k_1} \wedge \cdots \wedge \partial_{k_n} \to \partial_{k_1+\cdots+k_n}.$$

This is fairly natural with respect to the theory we've discussed. We have

$$Fun(Top_*, Top_*) \xrightarrow{\partial_*(-)} \partial_* - Bimod$$

which sends F to the bimodule with structure maps

$$\partial_n \wedge \partial_{k_1}(F) \wedge \cdots \wedge \partial_{k_n}(F) \mapsto \partial_{k_1+\cdots+k_n}(F).$$

Arone-Ching apply this additional structure to some known functors to obtain "quicker" proofs.

Fact 3.8. *The homology $H_*(\partial_*; \mathbb{Q})$ is an operad in $grVect_{\mathbb{Q}}$. Saying that V is an $H_*(\partial_*; \mathbb{Q})$-algebra is equivalent to saying that $\Sigma^{-1}V$ is a Lie algebra.*

This tells us what ∂_3 looks like:

$$\partial_3(id) = \bigvee_L S^{-2}$$

where L is all the ways one can bracket three elements in a Lie algebra. Concretely, we have

$$L = \{[x_1, [x_2, x_3]], [x_2, [x_1, x_3]], [x_3, [x_1, x_2]]\}/(\text{Jacobi identity}).$$

More basically, we have

$$\partial_2(id) = \bigvee_{\{[x_1,x_2],[x_2,x_1]\}/comm} S^{-1}.$$

Therefore if we want to compute $\pi_*(X)$, we have

$$\pi_*^s \mathbb{P}_{sLie}(X) \Rightarrow \pi_*(X)$$

where $\mathbb{P}_{sLie}$ is the free shifted Lie algebra in spectra. This looks "even more algebraic" than what we had before. This plays well with chromatic homotopy theory (e.g. computing the v_n-periodic unstable homotopy of X), which has been the subject of lots of recent work.