# INTRODUCTION TO MANIFOLD CALCULUS

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## 1. Introduction

Manifold calculus follows the same philosophy as homotopy calculus; however, neither is a special case of the other. Today's techniques will follow 1996 work of Weiss and Goodwillie.

Recall that for homotopy calculus, we worked with functors $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ was based topological spaces and $\mathcal{D}$ was based topological spaces or spectra. In manifold calculus, we will instead work with functors $\mathcal{O}(M)^{op} \to Top$, where $\mathcal{O}(M)$ is the category of open subsets of a manifold $M$ with morphisms given by inclusion.

**Definition 1.1.** We say that $F : \mathcal{O}(M)^{op} \to Top$ is *good* if

(1) $F$ takes isotopy equivalences to homotopy equivalences, and
(2) For any sequence of inclusions of open sets

$$U_0 \subseteq \cdots \subseteq U_i \subseteq \cdots ,$$

the map

$$F(U) \to holim_i(U_i)$$

is a homotopy equivalence.

All of the functors we work with today are good, so we won't worry too much about these conditions.

**Example 1.2.** The functors $Map(-, N) : U \mapsto Map(U, N)$, $Imm(-, N)$, and $Emb(-, N)$ are all good. Studying the functor $Emb(-, N)$ was the original motivation for these techniques.

## 2. Polynomial functors

Recall that a function $f$ is linear if $f(x + y) - f(x) - f(y) = 0$. We would like an analogy of this for functors.

**Definition 2.1.** We say that $F : \mathcal{O}(M)^{op} \to Top$ is *linear* if for all $V, W \subseteq M$, the total homotopy fiber of

$$\begin{array}{ccc} F(V \cup W) & \longrightarrow & F(V) \\ \downarrow & & \downarrow \\ F(W) & \longrightarrow & F(V \cap W). \end{array}$$

is contractible. Equivalently, this square is a homotopy pullback square.

As stated, this condition is hard to generalize to higher degrees. In other words, what is a polynomial functor of degree $k$?

**Definition 2.2.** We say that $F$ is *polynomial of degree* $\leq 1$ if for all $U \in \mathcal{O}(M)$ and for all disjoint, nonempty, closed subsets $A_0, A_1 \subseteq V$, the diagram

$$\begin{array}{ccc} F(U) & \longrightarrow & F(U - A_0) \\ \downarrow & & \downarrow \\ F(U - A_1) & \longrightarrow & F(U - (A_0 \cup A_1)) \end{array}$$

is homotopy cartesian.

Note that this recovers the previous definition if $W = U - A_0$ and $V = U - A_1$.

**Example 2.3.** The functors $Map(-, N)$ and $Imm(-, N)$ are both polynomial of degree $\leq 1$.

**Definition 2.4.** We say that $F$ is *polynomial of degree* $\leq k$ if for all $V \in \mathcal{O}(M)$, for all pairwise disjoint nonempty closed subsets $A_0, \ldots, A_k \subseteq V$, the $(k+1)$-cube $P(\underline{k+1}) \to Top$ formed by sending $S \mapsto F(V - \cup_{i \in S} A_i)$ is homotopy cartesian.

**Remark 2.5.** *This ends up coinciding with the definition of $k$-excisive.*

2.1. **Taylor tower.** Before we begin, we note that the standard notation in the manifold calculus literature differs from the notation in the homotopy calculus literature which was used in previous weeks.

**Definition 2.6.** Let $\mathcal{O}_k(M)$ be the subcategory of $\mathcal{O}(M)$ whose objects are the open subsets of $M$ which are diffeomorphic to at most $k$ open balls in $M$.

**Definition 2.7.** The *$k$-th stage of the Taylor tower* is defined by setting

$$T_k F(V) = holim_{U \in \mathcal{O}_k(V)} F(U).$$

This is the right homotopy Kan extension of the inclusion of $\mathcal{O}_k(M)$ into $\mathcal{O}(M)$.

We have natural transformations which are given by the maps of homotopy limits induced by the inclusion of $\mathcal{O}_k(V) \hookrightarrow \mathcal{O}(V)$. This gives

$$F(V) \simeq holim_{U \in \mathcal{O}(V)} F(U) \to holim_{U \in \mathcal{O}_k(V)} F(U) = T_k F(V),$$

where the first equivalence follows since $V$ is a final object in $\mathcal{O}(V)$. These maps fit into a tower of functors and natural transformations called the *Taylor tower*:

$$\begin{tikzcd}
& T_\infty F \arrow[d] \\
& \vdots \arrow[d] \\
& T_2 F \arrow[d] \\
F \arrow[r] \arrow[ur] & T_1 F.
\end{tikzcd}$$

where the maps $T_k F \to T_{k-1} F$ are induced by the inclusions $\mathcal{O}_{k-1}(V) \to \mathcal{O}_k(V)$. We obtain a map $F \to T_\infty F$ by taking the limit over $k$. We say that the tower *converges* if this map is an equivalence.

**Definition 2.8.** Let $L_k F = hofib(T_k F \to T_{k-1} F)$ be the $k$-th layer of this tower.

**Theorem 2.9** (Weiss, 1999)**.** (1) *$T_k F$ is polynomial of degree $\leq k$.*
(2) *If $F$ is polynomial of degree $\leq k$, then $F \to T_k F$ is a weak equivalence.*

## 3. Derivatives

It turns out that understanding the layers of the Taylor tower is equivalent to understanding the derivatives of $F$. More specifically, we need to understand the derivatives of $F$ at the empty set.

**Definition 3.1.** Let $B_1, \ldots, B_k$ be pairwise disjoint open balls in $M$. We can define a $k$-cube of spaces

$$S \mapsto F(\cup_{i \notin S} B_i).$$

The *$k$-th derivative of $F$ at $\emptyset$* is the total homotopy fiber of this cube. We will denote this by $F^{(k)}(\emptyset)$.

**Example 3.2.** For $k = 2$, take the total homotopy fiber of the square

$$\begin{tikzcd}
F(B_1 \cup B_2) \arrow[r] \arrow[d] & F(B_1) \arrow[d] \\
F(B_1) \arrow[r] & F(\emptyset).
\end{tikzcd}$$

We will work out some specific examples in the problem session.

How do the derivatives help us understand the layers of the Taylor tower?

**Proposition 3.3.** *Suppose that $F$ is good. If $F^{(k)}(\emptyset)$ is $c_k$-connected, then $L_k F(M)$ is $(c_k - km)$-connected.*

*For $U \subseteq M$, if $U$ has handle dimension $j$, then $L_k F(U)$ is $(c_k - kj)$-connected. Recall that, roughly speaking, $U$ has handle dimension $j$ if $j$ is the smallest integer such that $U$ admits a handlebody decomposition using handles of dimension $j$)*

## 4. The embeddings functor

**Fact 4.1.** *There is an equivalence*

$$T_1 Emb(V,N) \simeq T_1 Imm(V,N) \simeq Imm(V,N).$$

The second equivalence follows from the fact that $Imm(-,N)$ is polynomial of degree $\leq 1$, and the second fact follows from the fact that balls are contractible.

The Taylor tower then has the form

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_2 Emb(V,N) \arrow[d] \\
Emb(V,N) \arrow[r] \arrow[ur] & Imm(V,N).
\end{tikzcd}$$

This gives us a precise idea of how to lift an immersion to an embedding.

4.1. **Derivatives of embeddings.** Recall that we are applying the functor to disjoint open balls. In other words, we are computing the total homotopy fiber of cubes of spaces of the form $Emb(B_1 \cup \cdots \cup B_i, N)$. We know that each $B_\ell \simeq *$, so we can instead think of this as embedding of $i$ points in $n$:

$$Emb(B_1 \cup \cdots \cup B_i, N) \simeq Emb(\{x_1, \ldots, x_i\}, N) = Conf(i,N).$$

The right-hand side is the *configuration space of $i$ points in $N$.*

**Fact 4.2** (Fadell-Neuwirth)**.** *The projection maps*

$$Conf(i+1,N) \to Conf(i,N)$$

*are fibrations.*

**Example 4.3.** For $k=2$ and $N=\mathbb{R}^n$, we are considering the square

$$\begin{tikzcd}
Conf(2,\mathbb{R}^n) \arrow[r] \arrow[d] & Conf(1,\mathbb{R}^n) \arrow[d] \\
Conf(1,\mathbb{R}^n) \arrow[r] & Conf(0,\mathbb{R}^n).
\end{tikzcd}$$

The bottom-right corner is contractible, so the right-hand vertical fiber is contractible. The left-hand vertical fiber is $S^{n-1}$, so the total homotopy fiber is $S^{n-1}$.

It turns out that for larger $k$, we get wedges of spheres.

The analogous result for general manifolds is harder, but follows a similar proof.

**Proposition 4.4** (Weiss)**.** *In general, we have $Emb^{(k)}(\emptyset, N)$ is $(k-1)(n-2)$-connected.*

This tells us that $L_k Emb(M, N)$ is $(k-1)(n-2) - km = (k(n-m-2) - n + 2)$-connected, where $dim(M) = m$ and $dim(N) = n$. We can then apply results of Goodwillie to obtain the following result about the Taylor tower:

**Theorem 4.5** (Goodwillie-Weiss)**.** *The map*

$$Emb(M, N) \to T_k Emb(M, N)$$

*is $(k(n-m-2)-m+1)$-connected. In particular, the tower converges to $Emb(M, N)$ when $n > m + 2$.*

In homotopy calculus, we considered analyticity using homogeneous functors. There is an analogous notion in manifold calculus, but we used this approach today to see something new.