INTRODUCTION TO MANIFOLD CALCULUS

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1. INTRODUCTION

Manifold calculus follows the same philosophy as homotopy calculus; however, neither is a special case of the other. Today's techniques will follow 1996 work of Weiss and Goodwillie.

Recall that for homotopy calculus, we worked with functors $F : \mathcal{C} \to \mathcal{D}$ where \mathcal{C} was based topological spaces and \mathcal{D} was based topological spaces or spectra. In manifold calculus, we will instead work with functors $\mathcal{O}(M)^{op} \to Top$, where $\mathcal{O}(M)$ is the category of open subsets of a manifold M with morphisms given by inclusion.

Definition 1.1. We say that $F : \mathcal{O}(M)^{op} \to Top$ is good if

- (1) F takes isotopy equivalences to homotopy equivalences, and
- (2) For any sequence of inclusions of open sets

$$U_0 \subseteq \cdots \subseteq U_i \subseteq \cdots,$$

the map

$$F(U) \rightarrow holim_i(U_i)$$

is a homotopy equivalence.

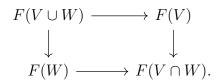
All of the functors we work with today are good, so we won't worry too much about these conditions.

Example 1.2. The functors $Map(-, N) : U \mapsto Map(U, N)$, Imm(-, N), and Emb(-, N) are all good. Studying the functor Emb(-, N) was the original motivation for these techniques.

2. Polynomial functors

Recall that a function f is linear if f(x+y) - f(x) - f(y) = 0. We would like an analogy of this for functors.

Definition 2.1. We say that $F : \mathcal{O}(M)^{op} \to Top$ is *linear* if for all $V, W \subseteq M$, the total homotopy fiber of



is contractible. Equivalently, this square is a homotopy pullback square.

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As stated, this condition is hard to generalize to higher degrees. In other words, what is a polynomial functor of degree k?

Definition 2.2. We say that F is polynomial of degree ≤ 1 if for all $U \in \mathcal{O}(M)$ and for all disjoint, nonempty, closed subsets $A_0, A_1 \subseteq V$, the diagram

$$F(U) \longrightarrow F(U - A_0)$$

$$\downarrow \qquad \qquad \downarrow$$

$$F(U - A_1) \longrightarrow F(U - (A_0 \cup A_1))$$

is homotopy cartesian.

Note that this recovers the previous definition if $W = U - A_0$ and $V = U - A_1$.

Example 2.3. The functors Map(-, N) and Imm(-, N) are both polynomial of degree ≤ 1 .

Definition 2.4. We say that F is polynomial of degree $\leq k$ if for all $V \in \mathcal{O}(M)$, for all pairwise disjoint nonempty closed subsets $A_0, \ldots, A_k \subseteq V$, the (k + 1)-cube $P(k + 1) \rightarrow Top$ formed by sending $S \mapsto F(V - \bigcup_{i \in S} A_i)$ is homotopy cartesian.

Remark 2.5. This ends up coinciding with the definition of k-excisive.

2.1. **Taylor tower.** Before we begin, we note that the standard notation in the manifold calculus literature differs from the notation in the homotopy calculus literature which was used in previous weeks.

Definition 2.6. Let $\mathcal{O}_k(M)$ be the subcategory of $\mathcal{O}(M)$ whose objects are the open subsets of M which are diffeomorphic to at most k open balls in M.

Definition 2.7. The *k*-th stage of the Taylor tower is defined by setting

$$T_k F(V) = holim_{U \in \mathcal{O}_k(V)} F(U).$$

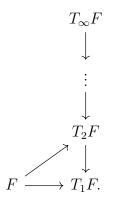
This is the right homotopy Kan extension of the inclusion of $\mathcal{O}_k(M)$ into $\mathcal{O}(M)$.

We have natural transformations which are given by the maps of homotopy limits induced by the inclusion of $\mathcal{O}_k(V) \hookrightarrow \mathcal{O}(V)$. This gives

$$F(V) \simeq holim_{U \in \mathcal{O}(V)} F(U) \to holim_{U \in \mathcal{O}_k(V)} F(U) = T_k F(V),$$

where the first equivalence follows since V is a final object in $\mathcal{O}(V)$. These maps fit into a tower of functors and natural transformations called the *Taylor tower*:

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where the maps $T_kF \to T_{k-1}F$ are induced by the inclusions $\mathcal{O}_{k-1}(V) \to \mathcal{O}_k(V)$. We obtain a map $F \to T_{\infty}F$ by taking the limit over k. We say that the tower *converges* if this map is an equivalence.

Definition 2.8. Let $L_kF = hofib(T_kF \to T_{k-1}F)$ be the k-th layer of this tower.

Theorem 2.9 (Weiss, 1999). (1) T_kF is polynomial of degree $\leq k$. (2) If F is polynomial of degree $\leq k$, then $F \to T_kF$ is a weak equivalence.

3. Derivatives

It turns out that understanding the layers of the Taylor tower is equivalent to understanding the derivatives of F. More specifically, we need to understand the derivatives of F at the empty set.

Definition 3.1. Let B_1, \ldots, B_k be pairwise disjoint open balls in M. We can define a k-cube of spaces

$$S \mapsto F(\cup_{i \notin S} B_i).$$

The k-th derivative of F at \emptyset is the total homotopy fiber of this cube. We will denote this by $F^{(k)}(\emptyset)$.

Example 3.2. For k = 2, take the total homotopy fiber of the square

$$F(B_1 \cup B_2) \longrightarrow F(B_1)$$

$$\downarrow \qquad \qquad \downarrow$$

$$F(B_1) \longrightarrow F(\emptyset).$$

We will work out some specific examples in the problem session.

How do the derivatives help us understand the layers of the Taylor tower?

Proposition 3.3. Suppose that F is good. If $F^{(k)}(\emptyset)$ is c_k -connected, then $L_kF(M)$ is $(c_k - km)$ -connected.

For $U \subseteq M$, if U has handle dimension j, then $L_kF(U)$ is $(c_k - kj)$ -connected. Recall that, roughly speaking, U has handle dimension j if j is the smallest integer such that U admits a handlebody decomposition using handles of dimension j)

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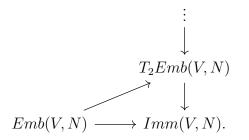
4. The embeddings functor

Fact 4.1. There is an equivalence

$$T_1Emb(V, N) \simeq T_1Imm(V, N) \simeq Imm(V, N).$$

The second equivalence follows from the fact that Imm(-, N) is polynomial of degree < 1, and the second fact follows from the fact that balls are contractible.

The Taylor tower then has the form



This gives us a precise idea of how to lift an immersion to an embedding.

4.1. Derivatives of embeddings. Recall that we are applying the functor to disjoint open balls. In other words, we are computing the total homotopy fiber of cubes of spaces of the form $Emb(B_1 \cup \cdots \cup B_i, N)$. We know that each $B_\ell \simeq *$, so we can instead think of this as embedding of *i* points in *n*:

$$Emb(B_1 \cup \cdots \cup B_i, N) \simeq Emb(\{x_1, \ldots, x_i\}, N) = Conf(i, N).$$

The right-hand side is the configuration space of i points in N.

Fact 4.2 (Fadell-Neuwirth). The projection maps

$$Conf(i+1, N) \to Conf(i, N)$$

are fibrations.

Example 4.3. For k = 2 and $N = \mathbb{R}^n$, we are considering the square

$$\begin{array}{ccc} Conf(2,\mathbb{R}^n) & \longrightarrow & Conf(1,\mathbb{R}^n) \\ & & & \downarrow \\ Conf(1,\mathbb{R}^n) & \longrightarrow & Conf(0,\mathbb{R}^n). \end{array}$$

The bottom-right corner is contractible, so the right-hand vertical fiber is contractible. The left-hand vertical fiber is S^{n-1} , so the total homotopy fiber is S^{n-1} .

It turns out that for larger k, we get wedges of spheres.

The analogous result for general manifolds is harder, but follows a similar proof.

Proposition 4.4 (Weiss). In general, we have $Emb^{(k)}(\emptyset, N)$ is (k-1)(n-2)-connected.

This tells us that $L_k Emb(M, N)$ is (k-1)(n-2) - km = (k(n-m-2) - n + 2)connected, where dim(M) = m and dim(N) = n. We can then apply results of
Goodwillie to obtain the following result about the Taylor tower:

Theorem 4.5 (Goodwillie-Weiss). The map

 $Emb(M, N) \rightarrow T_k Emb(M, N)$

is (k(n-m-2)-m+1)-connected. In particular, the tower converges to Emb(M, N) when n > m+2.

In homotopy calculus, we considered analyticity using homogeneous functors. There is an analogous notion in manifold calculus, but we used this approach today to see something new.