Part 1: Remembering Calculus

Sunday, March 3, 2019 12:45 PM

This talk: functors $f: T_* \to T_*$

$P_n f$ = n-excisive fn $\overset{\text{with map from } f}{\wedge}$ which is universal in $\mathrm{Ho}\,\mathrm{fun}(T_*, T_*)$

Thus $D_n f = \mathrm{Ho\,fib}(P_n f \to P_{n-1} f)$ is universal / htpy

$$\text{Have} \quad \left\{\begin{matrix} n\text{-homog} \\ \text{fns} \end{matrix}\right\} \underset{cr_n}{\overset{\longleftarrow}{\longrightarrow}} \left\{\begin{matrix} \text{symm } n\text{-multi} \\ \text{lin fns} \end{matrix}\right\} \quad \text{equiv on } \mathrm{Ho}(-)$$

(1)

$$\text{Assembly:} \quad \left\{\begin{matrix} \text{suitably} \\ \text{finite} \\ \text{symm lin} \end{matrix}\right\} \underset{\longleftarrow}{\xrightarrow{\text{coefficients}}} \{\text{spectra}\} \quad \text{equiv on } \mathrm{Ho}(-)$$

naive Σ_n

(3)

To follow $D_n f$ through, want f finitary: preserves filtered hocolims

The composite (1) $\to$ (3) defines $\partial_n f$ as image of $D_n f$ &

Goodwillie proves

$$\partial_n f \simeq \mathrm{hocolim}\, \Omega^{i_1} \Omega^{i_2} \cdots \Omega^{i_n} (cr_n f)(S^{i_1}, \ldots, S^{i_n})$$

In all:

$$\partial_* : \mathrm{fun}_{red}(T_*, T_*) \longrightarrow \mathrm{fun}(\Sigma, Sp)$$

which is nice on htpy categories

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Interlude: Categories of finite sets

Tuesday, March 5, 2019 10:51 AM

Def Let $\mathbb{I}$ = monoidal cat $\begin{cases} \text{obj} & \underline{n} = \{0, 1, \ldots, n-1\}, \quad n \geq 0 \\ \text{mor} & : \text{injections} : \mathrm{Aut}(n) = \Sigma_n \\ \text{product:} & \underline{n} \amalg \underline{m} = \underline{(n+m)} \end{cases}$

$\mathbb{N}$: $\begin{cases} \text{Same obj} \\ \text{mor} :: \text{inclusions } \underline{n-1} \hookrightarrow \underline{n} \quad \mathrm{Aut}(\underline{n}) = * \\ \text{product: } \amalg \end{cases}$

Σ : $\begin{cases} \text{Same obj} \\ \text{mor: bijections} \\ \amalg \end{cases}$

We're used to colims over $\mathbb{N}$, but sometimes we can extend to $\mathbb{I}$

Thm [Bökstedt]

Let $F : \mathbb{I} \to \mathcal{T}$, set $\mathbb{I}_k$ = full subcat on obs $\geq k$.

Let n_k = min over $\mathrm{Mor}(\mathbb{I}_k)$ of connectedness of maps

Then, if $n_k \to \infty$ as $k \to \infty$, the map

$$\underset{\mathbb{N}}{\mathrm{hocolim}}\, G \longrightarrow \underset{\mathbb{I}}{\mathrm{hocolim}}\, G$$

$$\operatorname{hocolim}_{\mathbb{N}} G \longrightarrow \operatorname{hocolim}_{\mathbb{I}} G$$

is an equiv.

We'll see in a bit why over $\mathbb{I}$ can be nicer

Recall (1) Symmetric sequence in cat $\mathcal{C}$:= functor $\Sigma \longrightarrow \mathcal{C}$

(2) Given $f, G : \Sigma \longrightarrow \mathcal{C}$, we define

$$(f \circ G)(\underline{n}) = \bigvee_{n_1 + \cdots + n_k = n} f(k) \wedge G(n_1) \wedge \cdots \wedge G(n_k)$$

(3) an operad is a monoid i.e. has a map

$$f \circ f \longrightarrow f$$

[Σ_n action on $f \circ f$ so that this is correct def of operad]

Part 2: Yeakel's model

Sunday, March 3, 2019 1:04 PM

Def (Yeakel's cr effect)

for $f: T_* \to T_*$, define $cr_n f: T_*^n \to T_*$

$$cr_1 f(X) = \mathrm{fib}(f(*) \to f(X))$$

$$cr_n f(X_1, \ldots, X_n) = cr_1^{(n)} \ldots cr_1^{(1)}(f \circ \sqcup_n)(X_1, \ldots, X_n)$$

n^{th} coordinate

$\sqcup_n(X_1, \ldots, X_n) = \bigvee_i X_i$

That is, cr_n gets fibers one direction at a time instead of total fiber of cube

Lemma $cr_n f$ has assembly maps in each variable. eg $n=2$

$$Z \wedge (cr_2 f)(X, Y) \longrightarrow (cr_2 f)(Z \wedge X, Y)$$

e.g.

$$\begin{array}{c} Z_1 \wedge Z_2 \wedge \cdots \wedge Z_n \wedge (cr_n f)(S^0, S^0, \ldots, S^0) \\ \downarrow \\ cr_n f(Z_1, Z_2, \ldots, Z_n), \end{array}$$

Σ_n-equivariant

No assertion of equiv or anything

Now similar to before, define derivs

$\boxed{\text{Main Def}}$ (Yeakel's derivs)

$$\partial_n f := \operatorname*{hocolim}_{(u_i) \in \mathcal{I}^n} \Omega^{u_1} \ldots \Omega^{u_n} cr_n f(\Sigma^{u_1} S^0, \ldots, \Sigma^{u_n} S^0)$$

Only diff from Goodwillie = over $\mathcal{I}$ not $\mathbb{N}$

Thm If f is Stably 1-excisive then the map

$$\partial_n^{\text{Goodwillie}} f \longrightarrow \partial_n f$$

induced by $N^n \longrightarrow I^n$ is an equivalence

Recall: analytic $\Longrightarrow$ stably n-excisive $\forall n$

$\Downarrow$ Taylor tower converges for highly connective spaces

stably n-excisive $\Updownarrow$ $E_n(c, \kappa)$ $\Updownarrow$ Preserves cartesian-ness of certain $n+1$ cubes

Pf idea Use Bökstedt's lemma + induction on n i.e. # coordinates we apply the lemma in

Connectiveness from stably 1-excisive hypothesis &

Lemma: If f is stably n-excisive, then $cr_n f$ is stably n-exc in each var

(Proven by hand with cubes)

Thm

$$\partial_* : \mathrm{Fun}(\mathcal{T}_*, \mathcal{T}_*)_{\text{red}} \longrightarrow \mathrm{Fun}(\Sigma, \mathcal{T}_*)$$

strictly reduced

is (lax) monoidal i.e. $\exists$ natural

$$\mu : \partial_* F \circ \partial_* G \longrightarrow \partial_*(F \circ G)$$

& map $\quad \varepsilon : \mathbb{1} \longrightarrow \partial_* \mathrm{id} \qquad \mathbb{1}(n) = \begin{cases} S^0 & n = 1 \\ * & \text{else} \end{cases}$

lax monoidal: this is not an iso, but it still takes monoids to monoids

i.e Monads $\downarrow$ to operad $\downarrow$

and algebras to modules over monad/operad

unit for $\mathrm{Fun}(\Sigma, \mathcal{T}_*)$

Pf for ε, need map

$$\varepsilon : \mathbb{1}(1)(X) \longrightarrow \partial_1(\mathrm{id})$$

$$S^0 \longrightarrow \operatorname*{hocolim}_{I} \Omega^n cr_1 \mathrm{Id}(S^n) \quad \Sigma_?$$

$$S^{\circ} \longrightarrow \operatorname{hocolim} \Omega^{-} cr_1 \mathrm{Id}(S^{-})$$

$$\mathrm{I}$$

$$\operatorname{hocolim}\left(S^0 \longrightarrow \Omega^1 S^1 \longrightarrow \Omega^2 S^2 \cdots\cdots \right)$$

with Σ_2 acting on $\Omega^2 S^2$

as first element

for μ: need maps

$$\partial_k F \wedge \partial_{j_1} G \wedge \cdots \wedge \partial_{j_k} G \longrightarrow \partial_j FG, \qquad j = \textstyle\sum j_i$$

E.G.

$$\partial_1 F \wedge \partial_1 G \longrightarrow \partial_1 FG$$

<u>fact</u> If $\mathcal{C}$ is a $\boxed{\text{nice}}$ topological category, and $H: \mathcal{C} \longrightarrow T_*$ is continuous, then there is

cotensored over T_*??

$$\alpha_H : Z \wedge \mathcal{C}(X) \longrightarrow \mathcal{C}(Z \wedge X)$$

Now,

$$\partial_1 F \wedge \partial_1 G = \left(\operatorname*{hocolim}_{U} \Omega^U cr_1 F(S^U)\right) \wedge \operatorname{hocolim} \Omega^V cr_1 G(S^V))$$

$$\Big\downarrow \alpha_{\mathrm{hocolim}} \text{ then } \alpha_{\Omega}$$

$$\operatorname*{hocolim}_{U,V} \Omega^{U \amalg V} (cr_1 F(S^U) \wedge cr_1 G(S^V))$$

$$\Big\downarrow \alpha_{cr_1 F} \text{ then } \alpha_{cr_1 G}$$

$$\operatorname*{hocolim}_{U,V} \Omega^{U \amalg V} cr_1 F(cr_1 G(S^{U \amalg V}))$$

$$\Big\downarrow (*)$$

$$\operatorname{hocolim} \Omega^{U \amalg V} cr_1 (F \circ G)(S^{U \amalg V})$$

$$\downarrow$$

$$\operatorname{hocolim}_{U,V} \Omega^{U \amalg V} cr_1(f\circ G)(S^{U \amalg V})$$

$$\Big\downarrow \amalg : \mathbb{I} \times \mathbb{I} \to \mathbb{I} \qquad (**)$$

$$\operatorname{hocolim}_{W} \Omega^{W} cr_1 f\circ G (S^W)$$

$$\|$$

$$\partial_1 f\circ G$$

Rmks (1) The map $(*)$: $cr_1 f(cr_1 G(x)) \longrightarrow cr_1(f\circ G)(x)$

is a special case of natural

$$\begin{array}{c} cr_k f \\ \swarrow \quad | \quad \searrow \\ cr_{j_1} G \quad \cdots \quad cr_{j_k} G \end{array} \longrightarrow cr_J f\circ G$$

which itself is from a map

$$f\circ cr_1 G \longrightarrow cr_1(f\circ G)$$

+ formal arguments

(2) $(**)$ is the reason to use $\mathbb{I}$:

while $\operatorname{hocolim}_{\mathbb{N}^2} \longrightarrow \operatorname{hocolim}_{\mathbb{N}}$ exists,

$\mathbb{I}$ keeps track better to let this be associative.

(3) The general case is formally similar using rmk)

<u>Thus</u> ∂_* is monoidal.

<u>Rmk</u> Can define spectrum $\underline{\partial_n f}$

$$(\underline{\partial_n f})_\ell = \underset{U_1, \dots, U_n}{\text{hocolim}}\, \Omega^{U_1} \cdots \Omega^{U_n} \Sigma^\ell cr_n f(S^{U_1}, \dots, S^{U_n})$$

<u>fact</u> • $\Omega^\infty \underline{\partial_n f} \simeq \partial_n f$

• $\underline{\partial_n f} \simeq \underline{\partial_n^{G} f}$ (Goodwillie def of Sp)

• $\underline{\partial_n} : \mathrm{Fun}(\mathcal{T}_*, \mathcal{T}_*) \longrightarrow \mathrm{Fun}(\Sigma, \mathrm{Sp})$

also monoidal

Part 3: Chain rule

Tuesday, March 5, 2019 9:30 PM

<u>Now</u>: ∂_n is always a spectrum.

- $\partial_* \mathrm{Id}$ is an operad:

$$\mathrm{Id} \circ \mathrm{Id} \longrightarrow \mathrm{Id} \rightsquigarrow \partial_* \mathrm{Id} \circ \partial_* \mathrm{Id} \longrightarrow \partial_* \mathrm{Id}$$

- $\partial_* f$ is a $\partial_* \mathrm{Id}$ bimodule from $f \circ \mathrm{Id} = \mathrm{Id} \circ f = f$.
- Can form diagram

$$\partial_* f \circ \partial_* G \leftrightarrows \partial_* f \circ \partial_* \mathrm{Id} \circ \partial_* G \cdots$$

Let $\partial_* f \underset{\partial_* \mathrm{Id}}{\circ} \partial_* G = \mathrm{hocolim}(\quad)$

<u>Thm</u> Let f, G be strictly reduced, finitary, analytic

$$\partial_* f \circ \partial_* G \longrightarrow \partial_* (f \circ G).$$

extends to an equivalence

$$\partial_* f \underset{\partial_* \mathrm{Id}}{\circ} \partial_* G \xrightarrow{\simeq} \partial_* (f \circ G)$$

<u>Pf</u>

Extension easy:

∂_* takes $\mathrm{Bar}(f, \mathrm{Id}, G) \longrightarrow f \circ G$

to $\mathrm{Bar}(\partial_* f, \partial_* \mathrm{Id}, \partial_* G) \longrightarrow \partial_* (f \circ G)$

Strategy: finitary is key. Start w/ G arbitrary,

$$F = H_X := \mathrm{Hom}(X, -) \qquad X = \text{finite CW}$$

Fiber seq (of functors)

$$\mathrm{Hom}(\vee S^{i+1}, -) \longrightarrow \mathrm{Hom}(X_{i+1}, -) \longrightarrow \mathrm{Hom}(X_i, -)$$

Lemma: ∂_K takes fiber seqs of functors to " of spectra $\boxed{S}$

Apply $\mathrm{Bar}(-, \partial_* \mathrm{Id}, \partial_* \mathrm{Id})$ to ∂_* above

(Bar preserves cofiber seqs)

Now:

$$\begin{array}{ccccc}
\partial_K H_{\vee S^{i+1}} \circ_{\partial_* \mathrm{Id}} \partial_* G & \longrightarrow & \partial_K H_{X_{i+1}} \circ_{\partial_* \mathrm{Id}} \partial_* G & \longrightarrow & \partial_K H_{X_i} \circ_{\partial_* \mathrm{Id}} \partial_* G \\
\downarrow (1) & & \downarrow (2) & & \downarrow (3) \\
\partial_K (H_{\vee S^{i+1}} \circ G) & \longrightarrow & \partial_K (H_{X_{i+1}} \circ G) & \longrightarrow & \partial_K (H_{X_i} \circ G)
\end{array}$$

Show by induction on i that (1) is an equiv,

thus (3) equiv $\Rightarrow$ (2) equiv

So induction proves case $F = H_X$

an ~~deriv~~ arbitrary T from representable
case

"Any cofibrant functor is equal to its left
Kan ext along identity"