No. 1

Date 3/20/19

N. Konovalov: Derivatives of identity (via Hilton-Milnor thm)

__Def.__ A __partition__ of $[\underline{n}]$ is an equivalence relation on $\underline{n}$.

eg $\underline{3} = \{1, 2, 3\}$ $\quad 1 \sim 2,\ 3.$

eg $\underline{5} = \underline{1}\ \ \underline{\underline{2}}\ \ \underline{3}\ \ \underline{\underline{4}}\ \ \underline{\underline{\underline{5}}}$

P_n = set of all partitions of $[\underline{n}]$

P_n is a poset: $\lambda_1 \leq \lambda_2$ if λ_1 is a refinement of λ_2, ie $x \sim_{\lambda_1} y \Rightarrow y \sim_{\lambda_2} y$.

P_n has initial and final objects: $\hat{0}, \hat{1}$.

discrete $\quad$ antidiscrete: $x \sim y\ \forall x, y$

$|P_n \setminus \{\hat{0}, \hat{1}\}| = \mathcal{P}_n$. space =: __partition complex__

__Thm__ (B. Johnson) $Id: Top_* \to Top_*$. Then

$$\partial_n(Id) \simeq D(\Sigma^\infty(\Sigma S \mathcal{P}_n)) \qquad (n \geq 1)$$

D: Spanier-Whitehead dual $\quad$ S: unred. susp.

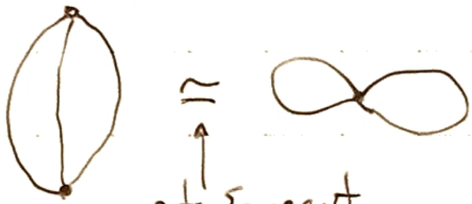
__Rmk.__ This is an equivalence of Borel Σ_n-spectra.

Ex. (n=1) doesn't work: $P_1 = \emptyset$.

Ex. (n=2) $P_2 = \emptyset$, $SP_2 = S^0$, $\Sigma SP_2 = S^1$.
$\Sigma^\infty S^1 = S^1$, $DS^1 = S^{-1}$,
(compare with previous exercises for $\partial_2 \mathrm{Id}$)

Ex (n=3) $P_3 = pt \sqcup pt \sqcup pt$ $\quad \Sigma_3$ permutes.

$SP_3 =$



not Σ_3-eqvt

$\Rightarrow D\Sigma^\infty \Sigma SP_3 \simeq S^{-2} \vee S^{-2}$

Rmk (1) $P_n \simeq \bigvee_{(n-1)!} S^{n-3}$ (non-eqvt'ly)

(2) P_n combinatorially accessible, understandable.

(3) $\Sigma SP_n \simeq \mathrm{Bar}(S, \mathrm{Comm}, S)(n)$

((n-1)! = dim free Lie alg on n elements)

No. 3

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<u>Def.</u> (<u>Samelson bracket</u>) G = H-space with htpy inverses.

$$G \times G \xrightarrow{[\,,\,]} G$$
$$(g,h) \longmapsto ghg^{-1}h^{-1}$$

$X \xrightarrow{f} G$, $\quad Y \xrightarrow{g} G$ $\quad$ g.l.e.s

$$\begin{tikzcd} X \times Y \arrow[r, "f \times g"] \arrow[d] & G \times G \arrow[r, "{[\,,\,]}"] & G \\ X \wedge Y \arrow[urr, dashed, "{[f,g]}"'] & & \end{tikzcd}$$

$[f,g]$ ← Samelson bracket

<u>Ex.</u> $X = S^m$, $Y = S^n$, $G = \Omega Z$.

$f : S^n \to \Omega Z \in \pi_{n+1} Z$

$g : S^m \to \Omega Z \in \pi_{m+1}(Z)$

$[f,g] : S^{m+n} \to \Omega Z \in \pi_{m+n+1}(Z)$

$\cong$

$\{f,g\}$ Whitehead product.

<u>Thm</u> (Hilton-Milnor) There is a weak equivalence

$$\prod_{w \in L_n}{}' \Omega\Sigma(X_w) \xrightarrow{\Phi} \Omega\Sigma(X_1 \vee \cdots \vee X_n)$$

for any $X_1, \ldots, X_n \in Top_*$, where

(1) Π' = colim of finite products

(2) L_n is a standard additive basis of the free Lie algebra on n elements:

$x_1, \dots, x_n, \quad [x_1, x_2], \dots, [x_i, x_j], \dots, [x_i, [x_j, x_k]], \dots$

(3) $w \in L_n$, $\quad X_w = ?$

eg $[x_1, [x_2, x_3]] = w \Rightarrow X_w = X_1 \wedge X_2 \wedge X_3$

(4) φ determined by $\varphi_w : \Omega\Sigma X_w \longrightarrow \Omega\Sigma(X_1 \vee \dots \vee X_n)$

φ_w determined as multiplicative extension

$$\psi_w : X_w \longrightarrow \Omega\Sigma(X_1 \vee \dots \vee X_n)$$

$$\left([Y, G]_{Top_*} \simeq [\Omega\Sigma Y, G]_{H\text{-space}} \right)$$

$$\begin{array}{l} \psi_w : X_w \longrightarrow \Omega\Sigma(X_1 \vee \dots \vee X_n) \\ \ \parallel \\ X_{i_1} \wedge \dots \wedge X_{i_k} \nearrow \end{array}$$

nested Samelson bracket of

$$X_{i_j} \longrightarrow \Omega\Sigma(X_1 \vee \dots \vee X_n)$$

Order of Samelson brackets prescribed by order of Lie brackets of w.

Rmk. ・) φ is not Σ_n-equivariant: L_n is not chosen equivariantly.

・) φ is not an H-space map.

No. 5

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Pf. (0) Reduction $n=2$: (formal) use

$$\Omega\Sigma X \times \Omega\Sigma\left(\bigvee_{k\geq 0} X^{\wedge k} \wedge Y\right) \longrightarrow \Omega\Sigma(X \vee Y)$$

$$[i_1, [i_1, \ldots, [i_1, i_2]] \ldots]].$$

(1) $H_*(\Omega\Sigma Z, F) \cong T(\tilde{H}_*(Z, F))$

tensor alg

(2) Algebraic part:

$$T(V \oplus W) \longleftarrow T(V) \otimes T\left(\bigoplus_{k\geq 0} V^{\otimes k} \otimes W\right)$$

$$[v_1, [v_2, \ldots, [v_k, w]\ldots]] \longleftarrow\!\!\!\longrightarrow v_1 \otimes v_2 \otimes \cdots \otimes v_k \otimes w$$

$$T(V \oplus W) = UL(V \oplus W)$$

univ. env. alg — free gr. Lie alg

$$0 \to K \to L(V \oplus W) \longrightarrow L(V) \longrightarrow 0$$

$$UL(V \oplus W) = UL(V) \otimes UK$$

$K = ?$

Lem. K is a free Lie alg

Pf. Subalg of free Lie is free Lie. //

Lem. K gen by $[v_1, [v_2, \ldots, [v_k, w] \ldots]]$.

Pf. exercise. //

(3) Topological part:

$$X \wedge Y \xrightarrow{[f,g]} G$$

$$H_*([f,g])(a \otimes b) \in H_*(G)$$ Hopf alg. since G is H-space

$(1 \otimes 2)\Delta f_*(a) \in H_*(G) \otimes H_*(G)$

$(2 \otimes 1)\Delta g_*(b) \in H_*(G) \otimes H_*(G)$ (*)

In H_*, $H_*([f,g])(a \otimes b)$: apply (*), swap middle factors, then multiply.

Rmk. $f_*(a)$, $g_*(b)$ primitive $\Rightarrow$ get Lie bracket. Usually not the case, though.

Reduce to primitively gen case by filtering by commutation ideal. Then done.

Next time: Non-equt derivatives

$$\partial_n(Id) \simeq \bigvee_{(n-1)!} S^{-(n-1)}$$

$$\|$$

$$\Sigma^{1-n}\partial_n(\Omega\Sigma) \qquad \partial_n(\Omega\Sigma) \simeq \bigvee_{(n-1)!} S^0.$$

·) $X_1, \dots, X_n \in Top_*$, $\mathcal{X} = n^{th}$ cube, $\mathcal{X}(S) = \bigvee_{i \notin S} X_i$.

$$\Omega\Sigma(\mathcal{X}(S)) = \prod_{w \in L_{n-|S|}}' \Omega\Sigma X_w$$

$$\text{tofib}(\Omega\Sigma\mathcal{X}) = \prod_{w \in L_n^0}' \Omega\Sigma X_w = \prod_{(n-1)!}(X_1 \wedge \dots \wedge X_n) \times (X_1^{\wedge 2} \wedge \dots \wedge X_n^{\wedge 2})$$