Linear algebra: Free Lie algebras

The purpose of this sheet is to fill the details in the algebraic part of the proof of the Hilton-Milnor theorem. By the way, the whole proof can be found in Neisendorfer's book "Algebraic Methods in Unstable Homotopy Theory."

Conventions: $\mathbf{k}$ is a field of characteristic $\neq 2$, all vector spaces are positively graded vector spaces over $\mathbf{k}$ and each graded component is finite dimensional, all associative algebras are connected and augmented. If A is an associative algebra, then $I(A)$ is its augmentation ideal (i.e. the kernel of the augmentation morphism).

Problem 1 (Graded Nakayama Lemma)**.** Let A be an associative algebra and let M be a graded A-module such that $M_n = 0$, for $n \ll 0$.

(a) If $I(A) \cdot M = M$, then $M = 0$;

(b) If $\mathbf{k} \otimes_A M = 0$, then $M = 0$.

Problem 2. Let A be an associative algebra and let V be a vector subspace of $I(A)$. Suppose that the augmentation ideal $I(A)$ is a free A-module generated by V. Prove that A is isomorphic to $T(V)$.

Problem 3. Let A be an associative algebra such that $\mathrm{Tor}_2^A(k,k) = 0$. Prove that A is isomorphic to the tensor algebra $T(V)$, where $V := I(A)/I(A)^2$ – the module of indecomposables.

Problem 4. Let L be a Lie algebra such that its universal enveloping associative algebra $U(L)$ is isomorphic to $T(V)$ for some V. Prove that L is isomorphic to the free Lie algebra $L(V)$. Hint: use a corollary of the Poincare-Birkhoff-Witt theorem which says that any Lie algebra L canonically injects into $U(L)$.

Problem 5. Let $L(V)$ be a free Lie algebra and let $L' \subset L(V)$ be a Lie subalgebra. Then L' is also free, that is, there is a vector space W such that $L' \cong L(W)$.

Problem 6. Denote by $h_V(t)$ the Poincare series of a vector space V, that is $h_V(t) := \sum t^i \dim V_i$. Prove

(a)
$$h_{T(V)}(t) = \frac{1}{1 - h_V(t)};$$

(b) Suppose that $0 \to L' \to L \to L'' \to 0$ is a split short exact sequence of Lie algebras and suppose that $L \cong L(V)$ is a free Lie algebra. Then $L' \cong L(W)$ for some W and the Poincare series of W is equal to
$$h_W(t) = 1 + (1 - h_V(t)) \cdot h_{UL''}(t).$$

Problem 7. Identify the kernel of the natural projection $L(V \oplus W) \to L(V)$.

Problem 8. Prove the algebraic Hilton-Milnor theorem. There is a natural isomorphism:

$$T(V \oplus W) \cong T(V) \otimes T\left(\bigoplus_{k \geq 0} V^{\otimes k} \otimes W \right).$$