N. Konovalov: Derivatives of identity II

No. 1

Date 3/27/19

Thm. (Johnson) $\partial_n(\mathrm{Id}) \underset{\Sigma_n\text{-eqvt}}{\simeq} \mathbb{D}(\Sigma^\infty \Sigma S P_n)$.

Recall. I = finite set.

$P(I)$ = poset of subsets of I

I-cube = functor $\mathcal{X}: P(I) \longrightarrow Top_*$.

$\mathrm{tfib}(\mathcal{X}) = \mathrm{holim}\left(\mathcal{X}(\emptyset) \to \lim\limits_{\substack{S\subset I \\ S\neq\emptyset}} \mathcal{X}(S)\right)$

Ex. $|I| = 1$. $P(I) = \{\emptyset, I\}$

I-cube $\mathcal{X}(\emptyset) \longrightarrow \mathcal{X}(I)$, $X \longrightarrow Y$

$|I|=2$, $P(I) = \{\emptyset, 0, 1, I\}$

$$\begin{tikzcd} \mathcal{X}(\emptyset) \arrow[r] \arrow[d] & \mathcal{X}(0) \arrow[d] \\ \mathcal{X}(1) \arrow[r] & \mathcal{X}(I) \end{tikzcd}.$$

$F : Top_* \longrightarrow Top_*$, $|I| = n$.

$cr_n F(X_1, \dots, X_n) := \mathrm{tfib}\left(S \longmapsto F\left(\bigvee_{i\notin S} X_i\right)\right)$

Ex. $n=1$, $cr_1 F = F$. (F based)

$n=2$, $cr_2 F(X_1, X_2) = \mathrm{tfib}\left(\begin{tikzcd} X_1 \vee X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd}\right)$

($F = \mathrm{id}$)

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<u>Fact</u>: $\Omega^\infty \partial_n F = \operatorname*{colim}_{k_1,\dots,k_n} \Omega^{k_1+\dots+k_n} cr_n(S^{k_1},\dots,S^{k_n})$

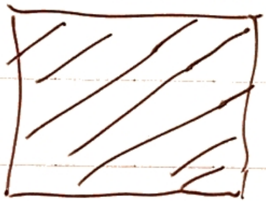
$cr_n(Id)(X_1,\dots,X_n) := \mathrm{tfib}\left(S \longmapsto \bigvee_{i\notin S} X_i\right)$

<u>Auxiliary cubes</u>: (1) I fixed finite set.

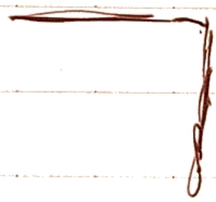
$S \longmapsto [0,1]^S = \{t\in[0,1]^I,\ t_i=0,\ i\notin S\}$.

(2) $S \longmapsto \partial_1 [0,1]^S = \{t\in[0,1]^S,$ s.t. some $t_i = 1$ for $i\in S\}$

<u>Ex</u>. $|I|=2$. $[0,1]^2 =$



$\partial_1 [0,1]^2 =$



<u>Lem</u>. (<u>exercise</u>)

$$\begin{tikzcd} \mathrm{tfib}(\mathcal{X}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Nat}([0,1]^\bullet, \mathcal{X}) \arrow[d] \\ pt \arrow[r] & \mathrm{Nat}(\partial_1[0,1]^\bullet, \mathcal{X}) \end{tikzcd}$$

from basept for $\mathcal{X}$

<u>Ex</u>. $|I|=1$.

$$\begin{tikzcd} X = \mathcal{X}(\emptyset) \arrow[r] & \mathcal{X}(I) = Y \\ {[0,1]^\emptyset} \arrow[u] \arrow[r] & {[0,1]} \arrow[u] \\ \{0\} \arrow[u, equal] \arrow[r, hook] & {[0,1]} \end{tikzcd}$$

$\mathrm{Nat}([0,1]^\bullet, \mathcal{X}) = Y^I \times_Y X$

$$\mathrm{Nat}\left(\partial, [0,1]^{\bullet}, \mathcal{X}\right) = \mathcal{Y}$$

$$\begin{tikzcd} X \arrow[r] & Y \\ \emptyset \arrow[u] \arrow[r, hook] & \{1\} \arrow[u] \end{tikzcd}.$$

$$\mathrm{tfib}(\mathcal{X}) = \left\{ \begin{array}{l} f_S : [0,1]^S \longrightarrow \mathcal{X}(S) \\ \forall S \subseteq I \end{array} \;\middle|\; \begin{array}{l} \text{compatible with} \\ S \subseteq T \\ f_S(t) = * \text{ if some } t_i = 1 \end{array} \right\}$$

$\forall i \in I$, have $\quad (|I| = n)$

$$\mathrm{tfib}(\mathcal{X}) \xrightarrow{p_i} \mathrm{Map}\left([0,1]^{I\setminus i}, \mathcal{X}(I\setminus i)\right)$$

$$\begin{tikzcd} \mathrm{tfib}(\mathcal{X}) \arrow[r, "p_1 \times \cdots \times p_n"] \arrow[dr, "\varphi"'] & \mathrm{Map}\left(\left([0,1]^{I\setminus i}\right)^I, \prod_{i\in I} \mathcal{X}(I\setminus i)\right) \arrow[d] \\ & \mathrm{Map}_*\left([0,1]^{n(n-1)}, \wedge\, \mathcal{X}(I\setminus i)\right) \end{tikzcd}.$$

$[0,1]^{n(n-1)}$ has coordinates $0 \le t_{ij} \le 1$, $\quad t_{ii} = 0 \quad 1 \le i,j \le n$

$$\varphi(\{f_S\}_{S\subseteq I})(t_{11}, \dots, t_{nn}) = f_{I\setminus 1}(t_{12}, \dots, t_{1n}) \wedge f_{I\setminus 2}(t_{21}, t_{23}, \dots, t_{2n}) \wedge \cdots \wedge f_{I\setminus n}(t_{n1}, \dots, t_{n(n-1)})$$

$cr_n(Id) = cr_n.$

$$\begin{tikzcd} cr_n(X_1, \dots, X_n) \arrow[r, "\varphi"] \arrow[dr, dashed, "\exists \varphi'"'] & \mathrm{Map}_*\left([0,1]^{n(n-1)}, \bigwedge_{i=1}^{n} X_i\right) \\ & \mathrm{Map}_*\left(\Delta_n, \bigwedge_{i=1}^{n} X_i\right) \arrow[u, hook] \end{tikzcd}$$

want to produce this

Def. (1) $Z \subset [0,1]^{n(n-1)}$, $Z = \{t \mid t_{ij} = 1 \text{ for some } i,j\}$

(2) $W_{ij} \subset [0,1]^{n(n-1)}$, $W_{ij} = \{t \mid t_{ik} = t_{jk} \ \forall\, 1 \le k \le n\}_{j<k}$

Lem. $f \in \mathrm{Map}_*([0,1]^{n(n-1)}, \wedge X_i)$ s.t. f is in the image of φ.

$\Rightarrow f(Z) = f(W_{ij}) = $ basept.

Pf. (W_{ij})

$$\begin{tikzcd}
 & {[0,1]^{n(n-1)}} \arrow[dl] \arrow[dr] & \\
 & W_{ij} \arrow[u, hook] \arrow[d, dotted] & \\
{[0,1]^{n-1}} \arrow[r, leftrightarrow] \arrow[d, "f_{I\setminus i}"] & {[0,1]^{n-2}} \arrow[r] \arrow[d, "f_{I\setminus\{i,j\}}"] & {[0,1]^{n-1}} \arrow[d, "f_{I\setminus j}"] \\
X_i & X_i \vee X_j \arrow[l] \arrow[r] & X_j
\end{tikzcd}$$

smash product in formula + maps to basept in X_i, X_j $\Downarrow$ maps to basept.

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Set $\Delta_n := I^{n(n-1)} / (Z \cup \bigcup_{i<j} W_{ij})$ and consider

$$\varphi : cr_n(X_1, \dots, X_n) \longrightarrow \mathrm{Map}_*\left(\Delta_n, \bigwedge_{i=1}^{n} X_i\right).$$

Fact. Let $F \xrightarrow{\varphi} G$ be a natural transformation of functors of n variables. Suppose that

$$\Omega\varphi_{\Sigma X_1, \dots, \Sigma X_n} : \Omega F(\Sigma X_1, \dots, \Sigma X_n) \longrightarrow \Omega G(\Sigma X_1, \dots, \Sigma X_n)$$

is $((n+1)k - C)$-connected whenever all X_i are k-connected. Then φ induces an equivalence after multilinearization.

$\Omega cr_n(\Sigma X_1, \dots, \Sigma X_n) = \text{tfib}\left(S \longmapsto \Omega\Sigma \bigvee_{i \notin S} X_i\right)$

Hilton-Milnor Thm $\Rightarrow$ product decomp of $\Omega\Sigma \bigvee_{i\notin S} X_i$. Then

$\Omega cr_n(\Sigma X_1, \dots, \Sigma X_n) = \prod'_{w \in L_n^0} \Omega\Sigma(X_w)$, $\quad L_n^0$ = set of words which use all $X_1, \dots, X_n$.

$\Rightarrow \pi_m(\Omega cr_n(\Sigma X_1, \dots, \Sigma X_n)) \cong \bigoplus_{(n-1)!} \pi_m(X_1 \wedge \dots \wedge X_n)$ for $0 \le m \le (n+1)(k+1)-1$

if each X_i is at least k-connected.

Other side: Δ_n $\quad$ (1) $\Delta_n \underset{\Sigma_n\text{-eqvt}}{\simeq} \Sigma SP_n$

(2) $\tilde\Delta_n \subseteq \Delta_n$, $\quad \tilde\Delta_n = \{t \mid t_{ij} = 0, j > 1\}$

$\tilde\Delta_n \simeq \bigvee_{(n-1)!} S^{n-1}$

(3) $\tilde\Delta_n \simeq \Delta_n$.

Pf. (1) $\Delta_n = I^{n(n-1)} / Z \cup \bigcup_{i<j} W_{ij} \simeq \Sigma\Big(\overset{\text{contractible}}{Z} \cup \overset{\text{contractible}}{\bigcup_{i<j} W_{ij}}\Big) \simeq \Sigma S\Big(\underbrace{Z \cap \big(\bigcup_{i<j} W_{ij}\big)}_{\text{cover } \{Z \cap W_{ij}\}}\Big)$

$\simeq \Sigma S P_n \Leftarrow$ partition ~ part. poset P_n (contractible, contractible intersection

(3) Similar

⋮

Prove Φ is surjective. Compute LHS using Samelson brackets. RHS $\Sigma\tilde\Delta_n \longrightarrow \Sigma\tilde\Delta_n \wedge S^1$.