Total fiber

The purpose of this sheet is to find a set-theoretic model for the total fiber of a cube diagram.

By the way, the content of the second talk can be found in Lecture 11 by Ben Knudsen given in Harvard, Spring 2018, on the seminar about unstable chromatic homotopy theory.

Definition. Let I be a finite set.

1. $P(I)$ is the poset of all subsets of I.
2. An I-cube of (based) topological spaces is a functor from $P(I)$ to Top_*.
3. Let $\mathscr{X}\colon P(I) \to \mathrm{Top}_*$ be an I-cube. The total fiber $\mathrm{tfib}(\mathscr{X})$ of $\mathscr{X}$ is defined by the following formula:
$$\mathrm{tfib}(\mathscr{X}) := \lim \Big(\mathscr{X}(\emptyset) \to \lim_{\emptyset \neq S \in P(I)} \mathscr{X}(S)\Big).$$
4. $[0,1]^\bullet\colon P(I) \to \mathrm{Top}$ is a non-based I-cube given by the rule:
$$[0,1]^S := \{t \in [0,1]^I \mid t_i = 0 \ \text{ for all } \ i \notin S\},$$
for all $S \in P(I)$.
5. $\partial_1[0,1]^\bullet$ is the following I-subcube of $[0,1]^\bullet$:
$$\partial_1[0,1]^S := \{t \in [0,1]^S \mid t_i = 1 \ \text{ for some } \ i \in S\},$$
for all $S \in P(I)$.
6. $\mathrm{Nat}(F,G)$ is a topological space of natural transformations between functors $F, G\colon P(I) \to \mathrm{Top}$.

Problem 1. Let $\mathscr{X}$ be an I-cube of based topological spaces.

1. Choose in a “natural” way a basepoint in the topological space $\mathrm{Nat}(\partial_1[0,1]^\bullet, \mathscr{X})$.
2. Show that the preimage of this basepoint under the canonical morphism
$$\mathrm{Nat}([0,1]^\bullet, \mathscr{X}) \to \mathrm{Nat}(\partial_1[0,1]^\bullet, \mathscr{X})$$
is homotopy equivalent to $\mathrm{tfib}(\mathscr{X})$.