Vortex Methods

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Integral Solutions to the Poisson Equation

\[ \nabla^2 \phi = \sigma \]

To solve the equation with a distributed source we integrate

\[ \phi(x) = \frac{-1}{4\pi} \int \frac{\sigma(x')}{r} \, dv' \]

The solution in a three-dimensional unbounded domain

The solution in a two-dimensional unbounded domain is

\[ \phi(x) = \frac{1}{2\pi} \int \sigma(x') \ln r \, dv' \]

\[ V^2 \psi = -\alpha \]

Solution

\[ \psi(x) = -\frac{1}{2\pi} \int \sigma(x') \ln r \, dv' \]

Velocity for 2D flow

\[ u(x) = \left( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right) = \frac{-1}{2\pi} \int o(x') \left( \frac{\partial \ln r}{\partial y}, \frac{\partial \ln r}{\partial x} \right) \, dv' \]

\[ u(x) = \frac{-1}{2\pi} \int \frac{o(x')(y - y')^2 + (x - x')^2}{r^2} \, dv' = \frac{1}{2\pi} \int \frac{k \times r}{r^2} o(x') \, dv' \]

since

\[ \frac{\partial \ln r}{\partial y} = \frac{1}{r} \ln \left( \frac{(y - y')^2 + (x - x')^2}{(x - x')^2 + (y - y')^2} \right) \]
Why Vortex Methods?

Helmholtz decomposition:
Any vector field can be written as a sum of
\[ u = \nabla \phi + \nabla \times \Psi \]
Take divergence
\[ \nabla \cdot u = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]
Take the curl
\[ \nabla \times u = \nabla \times (\nabla \times \Psi) = \omega \]
By a Gauge transform this can be written as
\[ \nabla^2 \Psi = -\omega \]

Vorticity

Helmholtz’s theorem:
Inviscid Irrotational flow remains irrotational

For incompressible flow with constant density and viscosity, taking the curl of the momentum equation yields:
\[ \frac{\partial \omega}{\partial t} + u \nabla \cdot \omega = ( \omega \cdot \nabla )u + \nabla^2 \omega \]

or:
\[ \frac{D\omega}{Dt} = (\omega \cdot \nabla )u + \nabla^2 \omega \]

Helmholtz’s theorem:
Inviscid Irrotational flow remains irrotational

Point Vortex Methods for 2D Flows

For incompressible unbounded flows, the motion everywhere, is completely determined by the vorticity. Thus, the evolution of the flow can be predicted by following the motion of the vorticity containing fluid ONLY.

\[ \frac{D\omega}{Dt} = 0 \]
\[ u(x) = \frac{1}{2\pi} \int_{\omega'} \frac{k \times r}{r'} \omega(x') \, da' \]

2D

\[ \frac{D\omega}{Dt} = (\omega \cdot \nabla )u \]
\[ u(x) = \frac{1}{4\pi} \int \frac{\omega \times r}{r'} \, dv' \]

3D
Euler Equation for two-dimensional inviscid incompressible flow

\[ \frac{d\omega}{dt} = 0 \]

The vorticity of each material particle is constant

\[ \nabla^2 \psi = -\omega \]

\[ u = \nabla \times (\psi k) \]

where \( K \) is the appropriate velocity kernel

The vorticity of each material particle is constant

\[ u(x) = \frac{1}{2\pi} \int K(x,x')\omega(x')dA' \]

For unbounded domain

\[ K(x,x') = \frac{k \times (x-x')}{|x-x'|^2} \]

Point vortices in an unbounded domain. The motion of each point is determined by

\[ \frac{dx_i}{dt} = \frac{1}{2\pi} \sum_{j=1}^{N} K(x_i,x_j)\Gamma_j \]

\[ \frac{d}{dt}(x_i,y_i) = \frac{1}{2\pi} \sum_{j=1}^{N} \Gamma_j \frac{(-y_j + y_i)(x_i - x_j))}{(x_i - x_j)^2 + (y_i - y_j)^2} \]

Generally, the point vortices are too singular to make a practical numerical method and the point vortices must be regularized. The simplest way is to find the velocity by

\[ \frac{d}{dt}(x_i,y_i) = \frac{1}{2\pi} \sum_{j=1}^{N} \Gamma_j \frac{(-y_j + y_i)(x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2 + \delta_i^2} \]

Use point vortices to approximate a smooth distribution of vorticity
Small viscosity can be added to vortex methods either by “random walk” or localized averaging.

In three-dimension it is necessary to account also for stretching and tilting of vortex lines, but the basic methodology still works.

The $N^2$ Problem

The $N^2$ problem:
To find the velocity of each vortex, it is necessary to sum over all the other vortices. This leads to large computational times when the number of vortices, $N$, is large.

Remedies:
- Grid based Vortex-In-Cell methods
- Fast summation methods

Vortex-In-Cell
Advect point vortices, but solve $\nabla^2 \psi = -\omega$ to find the velocities.

Fast summation method
A vortex far away from a group of vortices “sees” the group as a single large vortex.

By grouping the vortices together in an intelligent way, it is possible to reduce the operation count significantly.

Multipole expansion
For 2D flow we can rewrite the governing equations using complex numbers:

$$
\frac{dx}{dt} - i \frac{dy}{dt} = q' = \frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j}{z_j - z_i}
$$

$z = x + iy$
Vortex Methods for 3D Flows

For a thin filament:

\[ u(x) = -\frac{1}{4\pi} \int \frac{r}{(r^2 + \delta^2)^{3/2}} \frac{\partial x}{\partial \alpha} \, d\alpha \]

Discretization gives:

\[ u_j = -\frac{1}{4\pi} \sum_{i=1}^{N} \frac{x_j - x_{i+1}}{(x_j - x_{i+1})^2 + \delta^2} \times (x_{i+1} - x_j) \]
Often the vorticity is concentrated in a thin vortex sheet. Its strength is given by the integral of the vorticity across the sheet:

\[ \gamma = \int \omega \, ds = (u_i - u_j) \, s \]

Its velocity is given by

\[ u(x) = \frac{1}{2\pi} \int \frac{k \times r}{r^2} \omega(x') \, ds' = \frac{1}{2\pi} \int \frac{k \times r}{r^2} \gamma(s') \, ds' \]

Where \( P \) stands for the "principal value" of the integral.

To eliminate the need to consider a principal value integral and to stabilize short waves, the integral is usually regularized by adding a small numerical parameter

\[ u(x) = \frac{1}{2\pi} \int \frac{k \times r}{r^2 + \delta} \gamma(s') \, ds' \]

The sheet is usually discretized by replacing it with a row of point vortices

\[ \frac{d}{dt} \left( x_i, y_i \right) = \frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma \left( - (y_j - y_i) (x_i - x_j) \right)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \delta^2}} \]
Vortex Methods for Stratified Flows

Define the vortex sheet strength, the average velocity and the acceleration following an interface point:

\[ \gamma = (u_1 - u_2) \cdot s \quad U = \frac{1}{2} (u_1 + u_2) \]

By subtracting the tangential component of the inviscid Euler equations on either side of the interface

\[ \rho_1 \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) = -\nabla p - \rho_s g \]

We derive an equation for the evolution of the vortex sheet strength

\[ \frac{D\gamma}{Dt} + \gamma \frac{\partial U}{\partial s} \cdot s = 2.4 \left( \frac{D\gamma}{Dt} + \frac{1}{2} \frac{\partial U}{\partial t} \right) + 2.4 \gamma g \cdot s - \frac{2}{\rho_1 + \rho_2} \frac{\partial (p_1 - p_2)}{\partial s} \]

The evolution of the vortex sheet strength is found by

\[ \frac{D\gamma}{Dt} + \gamma \frac{\partial U}{\partial s} \cdot s = 2.4 \left( \frac{D\gamma}{Dt} + \frac{1}{2} \frac{\partial U}{\partial t} \right) + 2.4 \gamma g \cdot s - \frac{2}{\rho_1 + \rho_2} \frac{\partial (p_1 - p_2)}{\partial s} \]

Once the vortex sheet strength is known, the velocity is found by integrating over the sheet

\[ u(x) = \frac{1}{2\pi} \int \frac{k \times r}{r^2 + \delta} \gamma(s') \, ds' \]

The integration is sometimes replaced by the Vortex in Cell method

The Rayleigh Taylor Instability for different density ratios
Simulation of a 3D vortex ring colliding with an initially flat interface

Related Methods
- Panel and boundary integral method for flow over solid bodies
- Boundary Integral Methods for free surface flows
- Contour dynamics methods for “patches” of constant vorticity

There is no doubt that new solution strategies will continue to be developed. However, incremental advances of current approaches are likely to be the main vehicle for future advances in the computations of complex flows. The finite volume approach currently at the core of CFD has, in particular, proven to be exceedingly robust and versatile.