Computational Fluid Dynamics

The Equations Governing Fluid Motion

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Computational Fluid Dynamics

The rate of change of total mass in the control volume is given by:

\[
\frac{\partial}{\partial t} \int_V \rho \, dv
\]

Rate of change of total mass
Total mass
Control surface \( S \)
Control volume \( V \)

We now apply this statement to an arbitrary control volume in an arbitrary flow field.

Conservation of Mass

In general, mass can be added or removed:

The conservation law must be stated as:

Final Mass = Original Mass + Mass Added - Mass Removed

Conservation of Mass

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Conservation of Mass

Conservation of Mass

Conservation of Mass
To find the mass flux through the control surface, let's examine a small part of the surface where the velocity is normal to the surface.

Conservation of Mass

The sign of the normal component of the velocity determines whether the fluid flows in or out of the control volume. We will take the outflow to be positive: 

\[ u \cdot n < 0 \quad \text{Inflow} \]

The net in-flow through the boundary of the control volume is therefore:

Negative since this is net inflow

Integral over the boundary

Mass flow normal to boundary

Conservation of Mass Equation in Integral Form

\[ \frac{\partial}{\partial t} \int_V \rho \, dv = -\int_S \rho u \cdot n \, ds \]

Rate of change of mass

Net inflow of mass

Conservation of momentum

Momentum per volume: \( \rho u \)

Momentum in control volume: \( \int_V \rho u \, dv \)

\[ \text{Rate of increase of momentum} = \text{Net influx of momentum} + \text{Body forces} + \text{Surface forces} \]
The rate of change of momentum in the control volume is given by:

\[ \frac{\partial}{\partial t} \rho u dv \]

Rate of change
Total momentum

Select a small rectangle outside the boundary such that during time \( \Delta t \) it flows into the CV:

Body forces, such as gravity act on the fluid in the control volume.

The viscous force is given by the dot product of the normal with the stress tensor \( T \)

\( T = -pI + 2\mu D \)

For incompressible flows \( \nabla \cdot u = 0 \) and the stress tensor is

\[ T = -pI + 2\mu D \]

Whose components are:

\[ D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

The viscous flux of momentum

\[ T \cdot n ds \]

Total stress force

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Rate of change
Total momentum

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Total stress force
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Conservation of Momentum

Gathering the terms
\[ \frac{\partial}{\partial t} \rho u \, dv = -\frac{1}{2} \rho uu \cdot n \, ds - \frac{1}{2} T n \, ds + \int \rho f \, dv \]

Substitute for the stress tensor
\[ T = -p + 2\mu D \]

Rate of change of momentum
Net inflow of momentum
Total pressure
Total viscous force
Total body force

Conservation of energy

The energy equation in integral form
\[ \frac{\partial}{\partial t} \rho \left( e + \frac{1}{2} u^2 \right) dv = -\frac{1}{2} \rho \left( e + \frac{1}{2} u^2 \right) u \cdot n \, ds \]

Work done by body forces
Net work done by the stress tensor
Net heat flow

Differential Form of the Governing Equations

The Divergence or Gauss Theorem can be used to convert surface integrals to volume integrals
\[ \int_V \nabla \cdot a \, dv = \oint a \cdot n \, ds \]

Start with the integral form of the mass conservation equation
\[ \frac{\partial}{\partial t} \rho dv = -\int \rho u \cdot n \, ds \]

Using Gauss’s theorem
\[ \int \rho u \cdot n \, ds = \int \nabla \cdot (\rho u) \, dv \]

The mass conservation equations becomes
\[ \frac{\partial}{\partial t} \rho dv = -\int \nabla \cdot (\rho u) \, dv \]
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**Rearranging**

\[
\frac{\partial}{\partial t} \int_V \rho \, dv + \int_V \nabla \cdot (\rho u) \, dv = 0
\]

Since the control volume is fixed, the derivative can be moved under the integral sign.

\[
\int_V \frac{\partial \rho}{\partial t} \, dv + \int_V \nabla \cdot (\rho u) \, dv = 0
\]

**Expanding the divergence term:**

\[
\nabla \cdot (\rho u) = u \cdot \nabla \rho + \rho \nabla \cdot u
\]

The mass conservation equation becomes:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho + \rho \nabla \cdot u
\]

or

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot u = 0
\]

where

\[
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \cdot \nabla \rho
\]

**Convective derivative**

Using the mass conservation equation the advection part can be rewritten:

\[
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho uu) =
\]

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u =
\]

\[
\frac{\partial u}{\partial t} + \nabla \cdot (uu) + \rho \frac{Du}{Dt} = \rho \frac{Du}{Dt}
\]

\[
\frac{\partial u}{\partial t} + \nabla \cdot (uu)
\]

\[
= 0, \text{ by mass conservation}
\]

The momentum equation equation

\[
\frac{\partial \rho u}{\partial t} = \rho f + \nabla \cdot (T - \rho uu)
\]

Can therefore be rewritten as

\[
\frac{\rho Du}{Dt} = \rho f + \nabla \cdot T
\]

where

\[
\frac{\rho Du}{Dt} = \rho \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right)
\]
The energy equation can be converted to a differential form in the same way. It is usually simplified by subtracting the “mechanical energy equation”:

$$\rho \left( \frac{\partial u}{\partial t} + \rho u \cdot \nabla u \right) \cdot \nabla = -\rho \nabla \cdot \left( \nabla^2 u \right) + \rho u \cdot f + u \cdot (\nabla \cdot T)$$

The result is:

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = \rho f + \nabla \cdot T$$

The final result is

$$\rho \frac{\partial e}{\partial t} + p \nabla \cdot u = \Phi + \nabla \cdot k \nabla T$$

where

$$\Phi = \lambda (\nabla \cdot u)^2 + 2\mu \mathbf{D} \cdot \mathbf{D}$$

is the dissipation function and is the rate at which work is converted into heat.

Generally we also need:

$$p = p(e, \rho); \quad T = T(e, \rho);$$

and equations for $\mu, \lambda, k$

**Summary of governing equations**
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \]

\[ \rho \frac{\partial u}{\partial t} = \rho f + \nabla \cdot (T - \rho uu) \]

\[ \rho \frac{\partial (e + \frac{1}{2} u^2)}{\partial t} = \nabla \cdot (\rho (e + \frac{1}{2} u^2) u - uT + q) \]

\[ \frac{D\rho}{Dt} + \rho \nabla \cdot u = 0 \]

\[ \rho \frac{Du}{Dt} = \rho f + \nabla \cdot T \]

\[ \rho \frac{De}{Dt} = T \nabla \cdot u - \nabla \cdot q \]

Inviscid, compressible flows

\[ \mu = 0 \quad \text{and} \quad \lambda = 0 \]

\[ T = (\rho + \lambda V \cdot u)I + 2\mu D \]

\[ Dp \quad \rho \nabla \cdot u = 0 \]

\[ \rho \frac{Du}{Dt} = \rho f - \nabla p \]

\[ \rho \frac{De}{Dt} = -\nabla \cdot u \]

Incompressible flow
Incompressible flows:

\[
\frac{\partial}{\partial t} \rho + \vec{u} \cdot \nabla \rho = 0 \quad \nabla \cdot \vec{u} = 0
\]

Navier-Stokes equations (conservation of momentum)

\[
\frac{\partial}{\partial t} \rho \vec{u} = -\nabla P + \rho \vec{f} + \nabla \cdot \mu \nabla \vec{u}
\]

For constant viscosity

\[
\rho \frac{\partial}{\partial t} \vec{u} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla P + \rho \vec{f} + \mu \nabla^2 \vec{u}
\]

The 2D Navier-Stokes Equations for incompressible, homogeneous flow:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{\mu}{\rho} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\mu}{\rho} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

The pressure opposes local accumulation of fluid. For compressible flow, the pressure increases if the density increases. For incompressible flows, the pressure takes on whatever value necessary to prevent local accumulation:

- High Pressure: \( \nabla \cdot \vec{u} < 0 \)
- Low Pressure: \( \nabla \cdot \vec{u} > 0 \)

Increasing pressure slows the fluid down and decreasing pressure accelerates it.
Diffusion of fluid momentum is the result of friction between fluid particles moving at uneven speed. The velocity of fluid particles initially moving with different velocities will gradually become the same. Due to friction, more and more of the fluid next to a solid wall will move with the wall velocity.

\[ \partial \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \partial \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]

The momentum equations are:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{U}{L} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{U}{L} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]

The continuity equation is unchanged

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]