A Finite Difference Code for the Navier-Stokes Equations in Vorticity/Streamfunction Form

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Develop an understanding of the steps involved in solving the Navier-Stokes equations using a numerical method

Write a simple code to solve the “driven cavity” problem using the Navier-Stokes equations in vorticity form

Short discussion about why looking at the vorticity is sometimes helpful

Objectives

Computational Fluid Dynamics

• The Driven Cavity Problem
• The Navier-Stokes Equations in Vorticity/Streamfunction form
• Boundary Conditions
• The Grid
• Finite Difference Approximation of the Vorticity/Streamfunction equations
• Finite Difference Approximation of the Boundary Conditions
• Iterative Solution of the Elliptic Equation
• The Code
• Results
• Convergence Under Grid Refinement

Outline

The vorticity/streamfunction equations:

\[
\begin{align*}
-\frac{\partial}{\partial y} \left[ \frac{\partial \omega}{\partial x} + u \frac{\partial \omega}{\partial y} + v \frac{\partial \omega}{\partial y} \right] &= -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \\
\frac{\partial}{\partial x} \left[ \frac{\partial \omega}{\partial x} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} \right] &= -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \\
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} &= -\frac{1}{Re} \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) \\
\omega &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\end{align*}
\]

Solve the incompressibility conditions

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

by introducing the stream function

\[
u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}
\]

Substituting:

\[
\frac{\partial}{\partial x} \left( \frac{\partial \psi}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial \psi}{\partial x} \right) = 0
\]
Substituting \( u = \frac{\partial \psi}{\partial y} \) and \( v = -\frac{\partial \psi}{\partial x} \) into the definition of the vorticity \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) yields
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega
\]

The vorticity/streamfunction equations:

Substituting the advection-diffusion equation into the definition of the vorticity yields
\[
\frac{\partial \omega}{\partial t} + \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right)
\]

Recall the advection-diffusion equation
\[
\frac{\partial f}{\partial t} + \nu \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) = 0
\]

At the right and left boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \text{Constant}
\]

At the top and bottom boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
\]

Since the boundaries meet, the constant must be the same on all boundaries
\[
\psi = \text{Constant}
\]

At the top boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \text{Constant}
\]

At the bottom boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
\]

The normal velocity is zero since the streamfunction is a constant on the wall, but the zero tangential velocity must be enforced:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
\]

At the top boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \text{Constant}
\]

At the bottom boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
\]

At the top boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial y} = 0 \Rightarrow \psi = \text{Constant}
\]

At the bottom boundary:
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u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
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\]

At the bottom boundary:
\[
u = 0 \Rightarrow \frac{\partial \psi}{\partial x} = 0 \Rightarrow \psi = \text{Constant}
\]
Finite difference approximations

- \( \frac{\partial^2 f(x)}{\partial x^2} \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \)
- \( \frac{\partial f(x)}{\partial x} \approx \frac{f(x+h) - f(x-h)}{2h} + \cdots \)
- \( \frac{\partial^2 f(x)}{\partial t^2} \approx \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{\Delta t^2} + \cdots \)
- \( \frac{\partial f(t)}{\partial t} \approx \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} + \cdots \)

We start by laying down a discrete grid:

At the right and the left boundary:
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \quad \Rightarrow \quad \omega_{\text{wall}} = -\frac{\partial^2 \psi}{\partial x^2}
\]

Similarly, at the top and the bottom boundary:
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega \quad \Rightarrow \quad \omega_{\text{wall}} = -\frac{\partial^2 \psi}{\partial x^2}
\]

Use the notation developed earlier:

- \( f_{i,j} = f(x,y) \)
- \( f_{i+1,j} = f(x+h, y) \)
- \( f_{i,j+1} = f(x, y+h) \)
- \( f_{i+1,j+1} = f(x+h, y+h) \)
The vorticity at the new time is given by:

$$
\omega_{i,j}^{n+1} = \omega_{i,j}^n + \Delta t \left[ \left( \frac{\psi_{i,j+1}^n - \psi_{i,j-1}^n}{2h} \right) \left( \frac{\omega_{i,j+1}^n - \omega_{i,j-1}^n}{2h} \right) + \left( \frac{\psi_{i+1,j}^n - \psi_{i-1,j}^n}{2h} \right) \left( \frac{\omega_{i+1,j}^n - \omega_{i-1,j}^n}{2h} \right) + \frac{1}{Re} \left( \frac{\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i+1,j}^n + \omega_{i-1,j}^n - 4\omega_{i,j}^n}{h^2} \right) \right]
$$

The advection equation is:

$$
\frac{1}{h^2} \nabla^2 \psi_{i,j} + \frac{1}{Re} \frac{1}{h^2} \nabla^2 \omega_{i,j} = -\psi_{i,j}
$$

Consider the bottom wall ($j=1$):

Need vorticity on the boundary!
Solve by SOR

\[ \psi_{i,j+2} = \psi_{i,j+1} + \frac{\partial \psi_{i,j+1}}{\partial y} h + \frac{\partial^2 \psi_{i,j+1}}{\partial y^2} \frac{h^2}{2} + O(h^2) \]

Rewritten as

\[ \psi_{i,j+1} = 0.25 \left( \psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} + h^2 \omega_{i,j} \right) \]

Solve by SOR

\[ \psi_{i,j} = \beta \cdot 0.25 \left( \psi_{i+1,j}^{n+1} + \psi_{i-1,j}^{n+1} + \psi_{i,j+1}^{n+1} + \psi_{i,j-1}^{n+1} + h^2 \omega_{i,j}^{n+1} \right) \]

\[ + (1-\beta) \psi_{i,j}^{n+1} \]

Find RHS of vorticity equation

\[ \text{Solve for the wall vorticity:} \]

\[ \omega_{wall} = \left( \psi_{i,j+2} - \psi_{i,j+1} \right) \frac{2}{h} + U_{wall} \frac{2}{h} + O(h) \]

Limitations on the time step

\[ \frac{\nu \Delta t}{h^2} \leq \frac{1}{4} \]

\[ \frac{(\mu + 1) \nu |v| \Delta t}{\nu} \leq 2 \]
Computational Fluid Dynamics
The Code

```matlab
clf; nx = 9; ny = 9; MaxStep = 60; Visc = 0.1; dt = 0.02;
% resolution & governing parameters
MaxIt = 100; Beta = 1.5; MaxErr = 0.001;
% parameters for SOR iteration
sf = zeros(nx, ny); vt = zeros(nx, ny); w = zeros(nx, ny); h = 1.0 / (nx - 1);
t = 0.0;

for istep = 1:MaxStep,
% start the time integration
  for iter = 1:MaxIt,
    w = sf;
    % by SOR iteration
    for i = 2:nx-1;
      for j = 2:ny-1
        sf(i, j) = 0.25 * Beta * (sf(i+1, j) + sf(i-1, j) + sf(i, j+1) + sf(i, j-1) + h * h * vt(i, j)) + (1.0 - Beta) * sf(i, j);
      end;
    end;
    Err = 0.0;
    for i = 1:nx;
      for j = 1:ny,
        Err = Err + abs(w(i, j) - sf(i, j));
      end;
    end;
    if Err <= MaxErr, break, end
  end;
  vt(2:nx-1, 1) = -2.0 * sf(2:nx-1, 2) / (h * h);
  % vorticity on bottom wall
  vt(2:nx-1, ny) = -2.0 * sf(2:nx-1, ny-1) / (h * h) - 2.0 / h;
  % vorticity on top wall
  vt(1, 2:ny-1) = -2.0 * sf(2, 2:ny-1) / (h * h);
  % vorticity on right wall
  vt(nx, 2:ny-1) = -2.0 * sf(nx-1, 2:ny-1) / (h * h);
  % vorticity on left wall
  for i = 2:nx-1;
    for j = 2:ny-1
      w(i, j) = -0.25 * ((sf(i, j+1) - sf(i, j-1)) * (vt(i+1, j) - vt(i-1, j)) - (sf(i+1, j) - sf(i-1, j)) * (vt(i, j+1) - vt(i, j-1))) / (h * h) + Visc * (vt(i+1, j) + vt(i-1, j) + vt(i, j+1) + vt(i, j-1) - 4.0 * vt(i, j)) / (h * h);
    end;
  end;
  vt(2:nx-1, 2:ny-1) = vt(2:nx-1, 2:ny-1) + dt * w(2:nx-1, 2:ny-1);
  % update the vorticity
  t = t + dt
  subplot(121), contour(rot90(fliplr(vt))), axis('square');
  subplot(122), contour(rot90(fliplr(sf))), axis('square'); pause(0.01)
end;
```

Results:

![Streamfunction and Vorticity](image1)

17 by 17
Dt=0.01
D=0.1

Why is vorticity important?

Helmholtz decomposition:
Any vector field can be written as a sum of
\[ u = \nabla \phi + \nabla \times \Psi \]
Take divergence
\[ \nabla \cdot u = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]
Take the curl
\[ \nabla \times u = \nabla \times (\nabla \times \Psi) = \omega \]
By a Gauge transform this can be written as
\[ \nabla^2 \Psi = -\omega \]

For incompressible flow with constant density and viscosity, taking the curl of the momentum equation yields:
\[ \frac{\partial \omega}{\partial t} + u \nabla \cdot \omega = (\omega \cdot \nabla)u + \nabla^2 \omega \]
or:
\[ \frac{D\omega}{Dt} = (\omega \cdot \nabla)u + \nabla^2 \omega \]
Helmholtz’s theorem:
Inviscid Irrotational flow remains irrotational

In two-dimensions:
\[ \Psi = (0,0,\psi) \quad \omega = (0,0,\omega) \]
\[ \omega = \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} \]
or:
\[ \frac{D\omega}{Dt} = \nabla^2 \omega \quad \nabla^2 \psi = -\omega \]
Zero viscosity:
\[ \frac{D\omega}{Dt} = 0 \quad \text{The vorticity of a fluid particle does not change!} \]

Advection and diffusion—Boundary layers
Consider the steady state balance of advection and diffusion
\[ f(x=0) = 0 \quad U \frac{df}{dx} = f(x=L) = 1 \]

Governed by:
\[ U \frac{df}{dx} = D \frac{d^2f}{dx^2} \]

Solve this equation analytically
\[ \frac{df}{dx} = \frac{D}{U} \frac{d^2f}{dx^2} \quad \frac{df}{dx} \left( f - \frac{D}{U} \frac{d^2f}{dx^2} \right) = 0 \]
Integrate:
\[ f(x) = \frac{D}{U} \int_{x}^{L} \frac{df}{dx} = C_i \]

Integrate:
\[ f = \exp(Ux/D) \times \exp(C_i) + C_i \]

Boundary conditions
At \( x = 0 \):
\[ f(0) = 0 = \exp(C_i) + C_i \Rightarrow C_i = -\exp(C_i) \]
At \( x = L \):
\[ f(L) = 1 = \exp(UL/D) \times \exp(C_i) + C_i \]
\[ \Rightarrow 1 = \exp(UL/D) \times \exp(C_i) - \exp(C_i) \]
\[ \Rightarrow 1 = \exp(C_i) [\exp(U/L) - 1] \]
\[ \Rightarrow \exp(C_i) = \frac{1}{\exp(U/L) - 1} \]

Rearrange
\[ f - C_i = \frac{D}{U} \frac{df}{dx} \quad \frac{1}{f - C_i} \frac{df}{dx} = \frac{U}{D} \]
or
\[ \frac{df}{dx} = \frac{U}{D} \frac{1}{f - C_i} \]
Integrate
\[ \int \frac{df}{f - C_i} = \int \frac{U}{D} dx \quad \ln(f - C_i) = \frac{U}{D} x + C_2 \]
\[ f = \exp(Ux/D) \times \exp(C_i) + C_i \]

Scaling:
\[ \frac{df}{dx} = \frac{D}{U} \frac{d^2f}{dx^2} \]
\[ \frac{1}{\delta} \frac{D}{U} \frac{1}{\delta} = \delta \frac{DL}{UL} = \frac{L}{R} \]

Estimate the thickness of the "boundary Layer"
2D Solution of: \( U \frac{\partial f}{\partial x} = D \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \)

Objectives:

Develop an understanding of the steps involved in solving the Navier-Stokes equations using a numerical method

Write a simple code to solve the “driven cavity” problem using the Navier-Stokes equations in vorticity form

Short discussion about why looking at the vorticity is sometimes helpful