The Euler Equations

Ideal Gas:

The Euler Equations

The Rankine-Hugoniot conditions

Finding the eigenvalues for $A^T$
For the Euler equations:
\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \rho u + p \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ \rho E(u + p/\rho) \end{pmatrix} = 0
\]

The Rankine-Hugoniot conditions are:
\[
\begin{align*}
\alpha (\rho L - \rho R) &= (\rho u)_L - (\rho u)_R \\
\alpha ((\rho u)_L^2 + p)_L - ((\rho u)_R^2 + p)_R &= 0 \\
\alpha ((\rho u)_L(E + p/\rho))_L - ((\rho u)_R(E + p/\rho))_R &= 0
\end{align*}
\]

Consider the case \( \rho_L > \rho_R \):
Shock separates R and 2
Contact discontinuity separates 2 and 3
Expansion fan separates 3 and L

The Rankine-Hugoniot conditions give a nonlinear relation for the pressure jump across the shock \( P = \frac{\rho_L}{\rho_R} \)

\[
P = \frac{\rho_L}{\rho_R} \left[ 1 - \frac{(\gamma - 1)(c_L/c_R)(P - 1)}{\sqrt{2\gamma(\gamma + 1)(P - 1)}} \right]
\]

which can be solved by iteration.
The speed of the shock and velocity behind the shock are found using RH conditions:

\[ s_{\text{shock}} = u_s + c_s \left( \frac{\gamma - 1 + (\gamma + 1)\gamma}{2\gamma} \right) \]

The speed of the contact is:

\[ s_{\text{contact}} = u_c = u_l + \frac{2}{\gamma - 1} \left( \frac{p_R}{p_L} \right)^{\frac{\gamma - 1}{2\gamma}} \]

The left hand side of the fan moves with speed:

\[ s_L = u_l - c_s \]

The right hand side of the fan moves with speed:

\[ s_R = u_l + c_s \]

The Mach number is defined as the ratio of the local velocity over the speed of sound:

\[ Ma = \frac{u}{c} \]

The Mach number is defined as the ratio of the local velocity over the speed of sound:

\[ Ma = \frac{u}{c} \]

Test case:

Shocktube problem of G.A. Sod, JCP 27:1, 1978

- \( p_L = 10^5 \); \( \rho_L = 1.0 \); \( u_L = 0 \)
- \( p_R = 10^1 \); \( \rho_R = 0.125 \); \( u_R = 0 \)
- \( t_{\text{final}} = 0.005 \); \( \rho_{\text{final}} = 0.5 \)

Subsonic case

Exact Solution:

- Left uniform state
  - given:
    - \( p_L, \rho_L, u_L \)
  - \( u_i = u_l + \frac{2}{\gamma - 1} \left( \frac{p_R}{p_L} \right)^{\frac{\gamma - 1}{2\gamma}} \)
  - \( c_s = \sqrt{\frac{p_R}{\rho_L}} \)

- In the expansion fan (4)
  - \( \rho_i = \rho_l \left( \frac{1 + (s-s)_L}{c_s^2} \right) \)
  - \( u_i = u_l + \frac{2}{\gamma - 1} \left( \frac{s-s}{c_s^2} \right) \)
  - \( p_i = \rho_l \left( \frac{1 + (s-s)_L}{c_s^2} \right)^{\frac{\gamma - 1}{2\gamma}} \)

- Behind the contact (3)
  - \( p_i = p_L \)
  - \( u_i = u_L \)
  - \( \rho_i = \rho_L \left( \frac{p_R}{p_L} \right)^{\gamma/2} \)

Exact Solution:

- Computational Fluid Dynamics

Exact Solution:

- Computational Fluid Dynamics

Exact Solution:

- Computational Fluid Dynamics

Exact Solution:

- Computational Fluid Dynamics

Exact Solution:
Test case:

Shocktube problem of G.A. Sod, JCP 27:1, 1978

\[ p_L = 10^5; \quad \rho_L = 1.0; \quad u_L = 0 \]
\[ p_R = 10^4; \quad \rho_R = 0.01; \quad u_R = 0 \]
\[ t_{final} = 0.0025 \]
\[ x_0 = 0.5 \]

Supersonic case

For the fluid-dynamic system of equations (Euler equations):

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho E + pu \end{pmatrix} = 0
\]

where \( E = e + u^2 / 2 \); \( p = (\gamma - 1)e \rho \)

Add the artificial viscosity to RHS:

\[
\frac{\partial}{\partial x} \begin{pmatrix} 0 \\ \alpha h \rho \frac{\partial u}{\partial x} \\ \alpha h \rho \frac{\partial v}{\partial x} \end{pmatrix}
\]

Solutions of the 1D Euler equation using Lax-Wendroff

\[
F_{j+1}^{n+1} = F_j^n - \alpha h \rho \frac{\partial F_j^n}{\partial x}
\]

Lax Fredrich

\[
F_{j+1}^{n+1} = \frac{1}{2} \left( F_j^n + F_{j+1}^n \right) - \alpha \frac{h}{2} \frac{F_j^n - F_{j-1}^n}{h}
\]

Leap Frog

With an artificial viscosity term added to the corrector step

\[
F' = F - \alpha h \rho \frac{\partial F}{\partial x}
\]

Effect of \( \alpha \)

\[
\begin{array}{cccc}
\text{nx=128; } \\
\alpha = 0.5 & \alpha = 1.0 & \alpha = 1.5 & \alpha = 2.5
\end{array}
\]
\[
\frac{df}{dt} + \frac{\partial F}{\partial x} = 0
\]

Which can be written as

\[
\frac{df}{dt} + [A]\frac{\partial F}{\partial x} = 0; \quad [A] = \frac{\partial F}{\partial F}
\]

Define \( F = F^+ + F^- \) where \( [\lambda] = [\lambda^+] + [\lambda^-] \)

So that

\[
\frac{df}{dt} + \frac{\partial F^+}{\partial x} + \frac{\partial F^-}{\partial x} = 0
\]

Are the positive and negative eigenvalues of \( A \)

For nonlinear equation the splitting is not unique, different matrices can have the same eigenvalues.
Here we solve the one-dimensional Euler equation using the van Leer vector flux splitting

Van Leer

For 1D flow the fluxes are

For 2D flow the fluxes are

Several other splitting schemes are possible, such as:

Steger-Warming

Rewrite the flux terms in terms of Mach number:
\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0 \]

Solve by

\[ F^{n+1} = F^n - \frac{\Delta t}{2} \left( \frac{2c^2 - 1}{1 + \gamma} \right) \left( \frac{c^n - c^{n+1}}{c^{n+1} - c^n} \right) \]

Where

\[ F = \frac{\Delta t}{2} \left( \frac{2c^2 - 1}{1 + \gamma} \right) \left( \frac{c^n - c^{n+1}}{c^{n+1} - c^n} \right) \]

Effect of resolution

Shocktube problem of G.A. 

\( p_1 = 10^7; \quad p_2 = 1.0; \quad a_k = 0 \)

\( p_1 = 10^7; \quad p_2 = 0.125; \quad a_k = 0 \)

Final time: 0.005

Numerical solutions of the one-dimensional Euler equations

Upwind/flux splitting

Lax-Wendroff/artificial viscosity