Chapter 1

THE EQUATIONS OF FLUID DYNAMICS—DRAFT

The equations of fluid mechanics are derived from first principles here, in order to point out clearly all the underlying assumptions. The equations can take various different forms and in numerical work we will find that it often makes a difference what form we use for a particular problem.

We will work solely with the continuum theory of fluids, and thus use *conservation* principles, supplemented by *constitutive* assumptions about the nature of the fluids. The conservation principles are common to any material where the continuum hypothesis is valid but different constitutive hypothesizes apply to different materials. Expressing the basic principles of conservation of mass, momentum, and energy in mathematical form leads to the governing equations for fluid flow.

Here we derive the equations for fluid motion, with particular emphasize on incompressible flows.

1.1 General Flows

The principle of conservation of mass states that mass can not be created nor destroyed. Therefore, if we consider a volume fixed in space, V, then the change of mass inside this volume can only take place if mass flows in or out through the boundary of this volume, S.[1] Stated more precisely

$$\frac{d}{dt}\int_V \rho dv = -\oint_S \rho \mathbf{u}\cdot\mathbf{n} ds, \tag{1}$$

[1] In standard text books the fundamental laws are often stated for a volume of fluid moving with the fluid. In computational work the elementary volumes are usually stationary, therefore it is simpler to start with a stationary volume.

where $\mathbf{n}$ is the outward normal, ρ the density and $\mathbf{u}$ the velocity. Here, the left hand side is the rate of change of mass in the volume V and the right hand side represents in and out flow through the boundaries of V. Since the volume is fixed in space we can take the derivative inside the integral, and by applying the divergence theorem ($\int_V \nabla \cdot \mathbf{a} dv = \oint_S \mathbf{a} \cdot \mathbf{n} ds$) to the boundary fluxes on the right hand side we have

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) dv = 0. \tag{2}$$

This must hold for any arbitrary volume, no matter how small, and thus must also hold at a point, if the flow field is smooth. The partial differential equation expressing conservation of mass is therefore:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{3}$$

By using the definition of the substantial derivative

$$\frac{D()}{Dt} = \frac{\partial ()}{\partial t} + \mathbf{u} \cdot \nabla (), \tag{4}$$

the continuity equation can be rewritten as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{5}$$

The equation of motion is derived by assuming that Newton's 2^{nd} law of motion is valid for any arbitrary volume cut out of the fluid. Thus, the rate of change of momentum of a fixed volume is the net momentum flux across the boundaries of the volume plus the net forces acting on the volume. Therefore

$$\frac{d}{dt} \int_V \rho \mathbf{u} dv = -\oint_S \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) ds + \int_V \rho \mathbf{f} dv + \oint_S \mathbf{n} \mathbf{T} ds. \tag{6}$$

The first term on the right hand side is the momentum flux through the boundary of V, and the next terms are the body force $\mathbf{f}$ and surface forces, respectively. $\mathbf{T}$ is a symmetric stress tensor constructed in such a way that $\mathbf{nT}$ are the forces on the surface of V.

By the same arguments as applied to the mass conservation equation this must be valid at every point in the fluid, so

$$\frac{\partial \rho \mathbf{u}}{\partial t} = -\nabla \cdot (\rho \mathbf{u} \mathbf{u}) + \rho \mathbf{f} + \nabla \cdot \mathbf{T}. \tag{7}$$

Here, $\mathbf{uu}$ is a tensor whose ij-th component is $u_i u_j$. The nonlinear term can be written as

$$\nabla \cdot (\rho \mathbf{u} \mathbf{u}) = \rho \mathbf{u} \cdot \nabla \mathbf{u} + \mathbf{u} \nabla \cdot (\rho \mathbf{u}). \tag{8}$$

Using the definition of the substantial derivative and the continuity equation we can rewrite the above equation as

$$\rho \frac{D\mathbf{u}}{Dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{T}. \tag{9}$$

This is Cauchy's equation of motion and is valid for any continuous medium. For fluids like water, oil, and air (as well as many others that are generally referred to as Newtonian fluids) the stress may be assumed to be a linear function of the rate of strain, or

$$\mathbf{T} = (-p + \lambda\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu\mathbf{D} \tag{10}$$

where $\mathbf{I}$ is the unit tensor and $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the deformation tensor whose components are

$$D_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big). \tag{11}$$

Here, λ is the second coefficient of viscosity, and if Stoke's hypothesis is assumed to hold then $\lambda = -(2/3)\mu$. [2] For incompressible flows, the divergence of the velocity field $\nabla \cdot \mathbf{u}$, is identically zero, so any questions about the validity of Stokes hypothesis are irrelevant. Substituting the expression for the stress tensor into Cauchy's equation of motion results in

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{f} - \nabla p + \nabla(\lambda\nabla \cdot \mathbf{u}) + \nabla \cdot 2\mu\mathbf{D}. \tag{12}$$

Conservation of energy leads to

$$\frac{d}{dt}\int_V \rho(e+\frac{1}{2}u^2)dv = -\oint_S \rho(e+\frac{1}{2}u^2)\mathbf{u}\cdot\mathbf{n}ds + \int_V \mathbf{u}\cdot\rho\mathbf{f}dv + \oint_S \mathbf{n}\cdot(\mathbf{u}\mathbf{T}-\mathbf{q})ds \tag{13}$$

where $u^2 = \mathbf{u} \cdot \mathbf{u}$. The term on the left hand side is the sum of the internal and kinetic energy, the first term on the right hand side is flow of internal and kinetic energy across the boundaries, the second term represents work done by body forces, the third term is the work done by the stresses at the boundary (pressure and viscous) and $\mathbf{q}$ is the heat-flux vector. This equation can be simplified by using the momentum equation: Taking the dot product of the velocity with equation (9) gives

$$\rho\frac{\partial u^2/2}{\partial t} = -\rho\mathbf{u} \cdot \nabla u^2/2 + \rho\mathbf{u} \cdot \mathbf{f} + \mathbf{u} \cdot (\nabla \cdot \mathbf{T}) \tag{14}$$

for the *mechanical* energy. After using this equation to cancel terms in (13) and applying the same arguments as before we obtain the convective form of the energy equation

$$\rho\frac{De}{Dt} - \mathbf{T} \cdot \nabla\mathbf{u} + \nabla \cdot \mathbf{q} = 0. \tag{15}$$

This equation needs to be supplemented by constitutive equations for the specific fluids we are considering.

If we use the assumptions for the stress tensor made earlier (Newtonian fluids), and if we also assume that the flux of heat is proportional to the gradient of the temperature, T (this is called Fourie's law and k is the thermal conductivity)

$$\mathbf{q} = -k\nabla T, \tag{16}$$

[2]In several texts the discussion is based on the bulk viscosity $\kappa = 2/3\mu + \lambda$. Stoke's hypothesis is that the bulk viscosity vanishes.

and radiative heat transfer is negligible, then the energy equation takes the form

$$\rho \frac{De}{Dt} + p\nabla \cdot \mathbf{u} = \Phi + \nabla \cdot k\nabla T \tag{17}$$

Here, $\Phi = \lambda(\nabla \cdot \mathbf{u})^2 + 2\mu\mathbf{D} \cdot \mathbf{D}$ is called the dissipation function. It can be shown that Φ, which represents the rate at which work is converted into heat, is always greater or equal to zero. In general we also need an equation of state giving, say, pressure and temperature as a function of density and energy and equations for μ, λ and k.

$$p = p(e, \rho), \qquad T = T(e, \rho) \tag{18}$$

as well as equations for μ, λ and k.

1.2 Special Cases

In many practical applications, considerable savings can be achived by working with simplified form of the governing equations. Thus, the flows can be assumed to be invisicid if boundary layers do not play a large role, and incompressible if sound waves are not important.

1.2.1 Compressible, Inviscid Flows

In computations of compressible flows the conservative form is most frequently used, for reasons that we will discuss later. Furthermore, the viscosity is often taken to vanish. The equations are then generally written as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} + \frac{\partial \mathbf{F}}{\partial y} + \frac{\partial \mathbf{G}}{\partial z} = 0 \tag{19}$$

where

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix}; \mathbf{E} = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (\rho E + p)u \end{pmatrix}; \mathbf{F} = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (\rho E + p)v \end{pmatrix}; \mathbf{G} = \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (\rho E + p)w \end{pmatrix}. \tag{20}$$

If the gas is assumed to obey the perfect equation of state, then

$$P = \rho RT, \qquad e = c_v T, \qquad \text{and} \qquad h = c_p T. \tag{1.1}$$

Here R is the gas constant and $\gamma = c_p/c_v$. From these equations we find that

$$c_v = \frac{R}{\gamma - 1}; \qquad P = (\gamma - 1)\rho e; \qquad \text{and} \qquad T = \frac{(\gamma - 1)e}{R}. \tag{1.2}$$

$$\frac{d}{dt}\int_V \rho dv + \oint_S \rho \mathbf{u}\cdot\mathbf{n} ds = 0$$

$$\frac{d}{dt}\int_V \rho \mathbf{u} dv = \int_V \rho \mathbf{f} dv + \oint_S \Big(\mathbf{nT} - \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\Big) ds$$

$$\frac{d}{dt}\int_V \rho(e + \frac{1}{2}u^2) dv = \int_V \mathbf{u}\cdot\rho\mathbf{f} dv + \oint_S \mathbf{n}\cdot(\mathbf{uT} - \rho(e+\frac{1}{2}u^2)\mathbf{u} - \mathbf{q}) ds$$

Figure 1.1: The equations of fluid motion in integral form. The flux terms have been moved under the same surface integral as the stress terms.

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial\rho\mathbf{u}}{\partial t} = \rho\mathbf{f} + \nabla\cdot(\mathbf{T} - \rho\mathbf{uu})$$

$$\frac{\partial}{\partial t}\rho(e+\frac{1}{2}u^2) = \nabla\cdot\left(\rho(e+\frac{1}{2}u^2)\mathbf{u} - \mathbf{Tu} + \mathbf{q}\right)$$

Figure 1.2: The equations of fluid motion in conservative form.

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{u} = 0$$

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{f} + \nabla\cdot\mathbf{T}$$

$$\rho\frac{De}{Dt} = \mathbf{T}\cdot\nabla\mathbf{u} - \nabla\cdot\mathbf{q}$$

Figure 1.3: The equations of fluid motion in convective form.

1.2.2 Incompressible Flows

For an important class of flows the density of a material particle does not change as it moves with the flow. The equation of state is therefore

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 \tag{21}$$

and the continuity equation reduces to

$$\nabla \cdot \mathbf{u} = 0. \tag{22}$$

This states that the volume of any fluid element can not be changed and these flows are therefore usually called incompressible flows. By expanding the divergence in (2) and using the equation of state we find that

$$\oint_S \mathbf{u} \cdot \mathbf{n} ds = 0 \tag{23}$$

for incompressible flows. Notice that there is no restriction to constant density flows here, the density of a material particle can vary from one particle to the next one, but the density of each particle must stay constant. The density field is found by solving the equation of state, eq. (6), but if the density is constant everywhere that is not necessary.

The special case of incompressible fluids with constant density and viscosity is of considerable importance, and we will devote much space to that. The equations to be solved are then (22) and

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f} + \nu \nabla^2 \mathbf{u} \tag{24}$$

where $\nu = \mu/\rho$ is called the kinematic viscosity. This is the form usually seen in text books, but we note that the nonlinear term can be expressed in several forms:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot \mathbf{u}\mathbf{u} = \frac{1}{2} \nabla \mathbf{u} \cdot \mathbf{u} - \mathbf{u} \times (\nabla \times \mathbf{u}). \tag{25}$$

For numerical predictions equations (22) and (24) are actually somewhat awkward: We have evolution equations for each of the velocity component, and one expression for the relation between the velocities but really no separate equation for the pressure. This can be corrected by taking the divergence of the momentum equation and using the incompressibility conditions to eliminate the time derivative. This leads to

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) + \nabla \cdot \mathbf{f} \tag{26}$$

which replaces the incompressibility equation. We note the special role played by the pressure for incompressible fluid. Instead of being a thermodynamic function of (say) density and temperature, it is determined solely by the velocity field. Indeed, it will take on whatever value is necessary to make the flow divergence free. It is sometimes convenient—and we will use this extensively—to think of the pressure as projecting the velocity field into the space of incompressible functions. That is, we imagine the velocity first being predicted by (24) without the pressure, then we adjust the pressure in such a way as to enforce incompressibility and correct the velocities.

For incompressible fluids, where the density of a material particle is constant, the term expressing work done by compressing the fluid is absent. The internal heat generation due to viscous dissipation is frequently small, allowing us to ignore Φ. With these assumptions the energy equation reduces to

$$c_v \rho \frac{DT}{Dt} = \nabla \cdot k \nabla T \tag{27}$$

and is decoupled from the rest of the equations in the sense that while T depends on the velocity field, the velocity field does not depend on the temperature.

1.2.3 Vorticity form of the incompressible equations

The equations for incompressible flows can be rewritten, and integrated for special cases, by introducing two potentials such that the velocity is given by

$$\mathbf{u} = \nabla \phi + \nabla \times \boldsymbol{\Psi} \tag{28}$$

This is generally referred to as Helmholtz's decomposition, or as the fundamental theorem of vector analysis (for a proof see, e.g. []) and it can be shown, under fairly general conditions, satisfied in our case, that any vector field can be decomposed in this way. To derive equations for these potentials we take first the divergence of the definition (28) yielding

$$\nabla^2 \phi = \nabla \cdot \mathbf{u} = 0 \tag{29}$$

and then the curl

$$\nabla \times (\nabla \times \boldsymbol{\Psi}) = \nabla \times \mathbf{u} = \omega \tag{30}$$

Equation (30) may be simplified further if we note that it is really only the velocity field that we are interested in. We may therefore add a function to be $\boldsymbol{\Psi}$ as long as its curl remains the same. In particular, adding $\nabla \gamma$, where γ is a scalar function, produces the same velocity field. We use this to set $\nabla \cdot \boldsymbol{\Psi} = 0$. The reason why this is allowable can be seen as follows: Suppose $\nabla \cdot \boldsymbol{\Psi} \neq 0$. If we add $\nabla \gamma$ where $\nabla^2 \gamma = -\nabla \cdot \boldsymbol{\Psi}$ then obviously $\nabla \cdot (\boldsymbol{\Psi} + \nabla \gamma) = \nabla \cdot \boldsymbol{\Psi} - \nabla \cdot \boldsymbol{\Psi} = 0$ and if we work with $\boldsymbol{\Psi}' = \boldsymbol{\Psi} + \nabla \gamma$, this potential gives the same velocity field and is divergence free. Working with $\boldsymbol{\Psi}'$ (and dropping the prime) (30) becomes

$$\nabla^2 \boldsymbol{\Psi} = -\omega \tag{31}$$

where we have used that

$$\nabla \times (\nabla \times \boldsymbol{\Psi}) = \nabla(\nabla \cdot \boldsymbol{\Psi}) - \nabla^2 \boldsymbol{\Psi}. \tag{32}$$

This transform to a divergence free vector potential is called a Gauge Transform.

To obtain an equation for ω we take the curl of the Navier-Stokes's equations (24)

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u} + \omega(\nabla \cdot \mathbf{u}) - \nabla \times (\frac{\nabla p}{\rho}) + \nabla \times \Big(\frac{1}{\rho}\nabla \cdot \mu(\nabla \mathbf{u} + \nabla^T \mathbf{u})\Big) \quad (33)$$

For incompressible, constant density flow, this equation reduces to

$$\frac{D\omega}{Dt} = (\omega \cdot \nabla)\mathbf{u} + \nu\nabla^2\omega. \quad (34)$$

When the flow is two-dimensional, both the vector stream function, and the vorticity vector have only one non-zero component,

$$\mathbf{\Psi} = (0, 0, \psi), \qquad \omega = (0, 0, \omega) \quad (35)$$

where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (36)$$

The right hand side of (9) is then identically zero, and we have to solve

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega, \qquad \nabla^2\psi = -\omega. \quad (37)$$

The velocity is

$$\mathbf{u} = \nabla \times \psi\hat{k} = \nabla\psi \times \hat{k} \quad (38)$$

or

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}. \quad (39)$$

The steam function can be eliminated in favor of the velocity components by taking the curl of equation (37). This results in an elliptic equation relating the velocity components to the curl of the vorticity

$$\nabla^2\mathbf{u} = -\nabla \times \omega, \quad (40)$$

or

$$\nabla^2 u = -\frac{\partial \omega}{\partial y}; \qquad \nabla^2 v = \frac{\partial \omega}{\partial x} \quad (41)$$

in two dimension.

We have seen above that there are three basic form of the equations that may be used as the basis of an numerical approximation. Which of these form is best depends on the dimensions of the problem, the boundary conditions, and what the programmer is used to. The number of equations is obviously an important factor and the following table summarizes that.

Method	Dimensions	# of elliptic eqs.	# of advection eqs.
$\mathbf{u} - p$	2	1	2
$\mathbf{u} - p$	3	1	3
$\psi - \omega$	2	1	1
$\mathbf{\Psi} - \omega$	3	3	3
$\omega - \mathbf{u}$	2	2	1
$\omega - \mathbf{u}$	3	3	3

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{f} + \nu \nabla^2 \mathbf{u}$$

$$\nabla^2 p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$$

Primitive variables

$$\frac{\partial \omega}{\partial t} + (\nabla \times \mathbf{\Psi}) \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} = \nu \nabla^2 \omega$$

$$\nabla^2 \mathbf{\Psi} = -\omega$$

Streamfunction-vorticity variables

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega - \omega \cdot \nabla \mathbf{u} = \nu \nabla^2 \omega$$

$$\nabla^2 \mathbf{u} = -\nabla \times \omega$$

Velocity-vorticity variables

Figure 1.4: Three forms of the incompressible Navier-Stokes equations

Although the number of equations does vary, depending on which formulation is used, their fundamental structure remains the same. In all cases we have one or more advection-diffusion equation describing the evolution of certain quantities with time and one or more elliptic equation, representing a quantity that at each instant has equilibrated over the whole domain. The nature of the boundary conditions does though change from one formulation to the next.