For most engineering problems it is necessary to deal with complex geometries, consisting of arbitrarily curved and oriented boundaries.

Outline

How to deal with irregular domains
- Overview of various strategies
- Boundary-fitted coordinates
  - Navier-Stokes equations in vorticity form
  - Navier-Stokes equations in primitive form
- Grid generation for body-fitted coordinates
  - Algebraic methods
  - Differential methods

Various Strategies for Complex Geometries (Primarily to treat solid boundaries)
- Staircasing
- Boundary-fitted coordinates
- Immersed boundary method (no grid change)
- Unstructured grids: triangular or tetrahedral
- Adaptive mesh refinement (AMR)

Staircasing
Approximate a curved boundary by a the nearest grid lines

Boundary-Fitted Coordinates
• Coordinate mapping: transform the domain into a simpler (usually rectangular) domain.
• Boundaries are aligned with a constant coordinate line, thus simplifying the treatment of boundary conditions
• The mathematical equations become more complicated

Solving for the derivatives

\[
\frac{\partial f}{\partial x} = \frac{1}{J} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x} \right)
\]
\[
\frac{\partial f}{\partial y} = \frac{1}{J} \left( \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y} \right)
\]

where
\[
J = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi}
\]
is the Jacobian.

First-Order Derivatives

Change of variables
\[
f(x, y) = f(x(\xi, \eta), y(\xi, \eta)) = f(\xi, \eta)
\]

Note: The equations will be discretized in the new grid system \((\xi, \eta)\). Therefore, it is important to end up with terms like \(\partial x / \partial \xi\), not \(\partial \xi / \partial x\).

We want to derive expressions for \(\partial f / \partial x\), \(\partial f / \partial y\) in the mapped coordinate system.

Using the chain rule:
\[
\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \xi}
\]
\[
\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \eta}
\]
A short-hand notation:

\[
\begin{align*}
  f_x &= \frac{\partial f}{\partial x}; \\
  f_y &= \frac{\partial f}{\partial y}; \\
  f_\xi &= \frac{\partial f}{\partial \xi}; \\
  f_\eta &= \frac{\partial f}{\partial \eta}, \\
  y_\xi &= \frac{\partial y}{\partial \xi}; \\
  y_\eta &= \frac{\partial y}{\partial \eta}; \\
  x_\xi &= \frac{\partial x}{\partial \xi}; \\
  x_\eta &= \frac{\partial x}{\partial \eta}.
\end{align*}
\]

Since:

\[
\begin{align*}
f &= \frac{1}{J} \left( f_\eta y_\xi - f_\xi y_\eta \right), \\
f &= \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right)
\end{align*}
\]

Rewriting in short-hand notation:

\[
\begin{align*}
f_x &= \frac{1}{J} \left( f_\eta y_\xi - f_\xi y_\eta \right), \\
f_y &= \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right)
\end{align*}
\]

where

\[
J = x_\xi y_\eta - x_\eta y_\xi
\]

is the Jacobian.

These relations can also be written in conservative form:

\[
\begin{align*}
f_x &= \frac{1}{J} \left[ \left( f_\eta y_\xi - f_\xi y_\eta \right) - \right] \\
f_y &= \frac{1}{J} \left[ \left( f_\xi x_\eta - f_\eta x_\xi \right) - \right]
\end{align*}
\]

Since:

\[
f &= \frac{1}{J} \left[ \left( f_\eta y_\xi - f_\xi y_\eta \right) + \left( f_\xi x_\eta - f_\eta x_\xi \right) \right] = \frac{1}{J} \left[ f_\xi y_\eta - f_\eta x_\xi \right]
\]

And similarly for the other equation.

Second-Order Derivatives

The second derivatives is found by repeated application of the rules for the first derivative:

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \right) x_\xi \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \right) x_\eta
\end{align*}
\]

Similarly

\[
\begin{align*}
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{J} \left[ \left( f_\xi x_\eta - f_\eta x_\xi \right) \right] x_\xi - \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \frac{y_\eta}{y_\xi} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{1}{J} \left[ \left( f_\xi x_\eta - f_\eta x_\xi \right) \right] x_\eta - \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \frac{x_\xi}{x_\eta}
\end{align*}
\]

Adding

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} =
\]

and

\[
\frac{\partial^2 f}{\partial x^2} = \frac{1}{J} \left[ \left( f_\xi x_\eta - f_\eta x_\xi \right) \right] x_\xi - \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \frac{y_\eta}{y_\xi} \\
\frac{\partial^2 f}{\partial y^2} = \frac{1}{J} \left[ \left( f_\xi x_\eta - f_\eta x_\xi \right) \right] x_\eta - \frac{1}{J} \left( f_\xi x_\eta - f_\eta x_\xi \right) \frac{x_\xi}{x_\eta}
\]

yields an expression for the Laplacian:

\[
\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\]
\[ \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{J^2} \left[ \frac{\partial}{\partial \xi} \left( q_1 f_\xi - q_2 f_\eta \right) + \frac{\partial}{\partial \eta} \left( q_2 f_\xi - q_1 f_\eta \right) \right] + \frac{1}{J^4} \left[ -q_2 f_{\xi \xi} + q_1 f_{\eta \eta} + q_2 f_{\xi \xi} - q_1 f_{\eta \eta} \right] \]

where
\[ q_1 = x_\xi^2 + y_\eta^2 \]
\[ q_2 = x_\xi y_\eta + y_\xi y_\eta \]
\[ q_3 = x_\eta^2 + y_\xi^2 \]

Expanding the derivatives yields
\[ \nabla^2 f = \frac{1}{J} \left( q_1 f_{\xi \xi} - 2q_2 f_{\xi \eta} + q_3 f_{\eta \eta} \right) + \left( \nabla^2 \right) f_\xi + \left( \nabla^2 \right) f_\eta \]

where
\[ \nabla^2 \xi = \frac{1}{J} \left[ f(x_\xi x_\xi - x_\eta x_\eta - x_\xi y_\eta + y_\xi y_\eta) - q_1 f_\xi - q_2 f_\eta \right] \]
\[ \nabla^2 \eta = \frac{1}{J} \left[ f(x_\xi x_\xi - x_\eta x_\eta - x_\xi y_\eta + y_\xi y_\eta) + q_2 f_\xi - q_1 f_\eta \right] \]

\[ J_\xi = x_\xi x_\xi + x_\eta x_\eta - x_\xi y_\eta - y_\xi y_\eta \]
\[ J_\eta = x_\xi x_\xi + x_\eta x_\eta - x_\xi y_\eta - y_\xi y_\eta \]

Thus:
\[ f_\xi = \frac{1}{J} \left( f_\xi y_\eta - f_\eta x_\eta \right) \]
\[ f_\eta = \frac{1}{J} \left( f_\xi y_\xi - f_\xi x_\eta \right) \]

Putting them together, it can be shown that (prove it!)

\[ \nabla^2 \xi = \frac{1}{J} \left[ f \left( x_\xi y_\xi - y_\xi y_\xi \right) - 2q_1 f_\xi \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( x_\xi y_\eta - y_\xi y_\eta \right) \right] \]
\[ \nabla^2 \eta = \frac{1}{J} \left[ f \left( x_\eta y_\eta - y_\eta y_\eta \right) - 2q_1 f_\eta \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( y_\xi y_\eta - x_\eta y_\eta \right) \right] \]

Putting them together, it can be shown that (prove it!)

\[ \nabla^2 \xi = \frac{1}{J} \left[ f \left( x_\xi y_\xi - y_\xi y_\xi \right) - 2q_1 f_\xi \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( x_\xi y_\eta - y_\xi y_\eta \right) \right] \]
\[ \nabla^2 \eta = \frac{1}{J} \left[ f \left( x_\eta y_\eta - y_\eta y_\eta \right) - 2q_1 f_\eta \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( y_\xi y_\eta - x_\eta y_\eta \right) \right] \]

Putting them together, it can be shown that (prove it!)

\[ \nabla^2 \xi = \frac{1}{J} \left[ f \left( x_\xi y_\xi - y_\xi y_\xi \right) - 2q_1 f_\xi \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( x_\xi y_\eta - y_\xi y_\eta \right) \right] \]
\[ \nabla^2 \eta = \frac{1}{J} \left[ f \left( x_\eta y_\eta - y_\eta y_\eta \right) - 2q_1 f_\eta \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( y_\xi y_\eta - x_\eta y_\eta \right) \right] \]

Putting them together, it can be shown that (prove it!)

\[ \nabla^2 \xi = \frac{1}{J} \left[ f \left( x_\xi y_\xi - y_\xi y_\xi \right) - 2q_1 f_\xi \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( x_\xi y_\eta - y_\xi y_\eta \right) \right] \]
\[ \nabla^2 \eta = \frac{1}{J} \left[ f \left( x_\eta y_\eta - y_\eta y_\eta \right) - 2q_1 f_\eta \left( x_\xi y_\eta - y_\xi y_\eta \right) + q_1 \left( y_\xi y_\eta - x_\eta y_\eta \right) \right] \]
The Navier-Stokes equations in vorticity form become:

\[ \frac{\partial \omega}{\partial t} + \frac{1}{J} \left( \psi \omega_x - \psi_x \omega \right) = \frac{1}{\tau} \left( q \omega_z - 2 q \omega_{xz} + q \omega_{zx} \right) + \frac{1}{\tau} \left( \nabla \nabla^2 \xi \right) \psi_x + \frac{1}{\tau} \left( \nabla \nabla^2 \eta \right) \psi_\eta = -\omega \]

\[ q_x = x_x + x_\eta \]
\[ q_y = x_x + x_\eta \]
\[ q_z = x_x + x_\eta \]

Boundary Conditions

Lower wall (\eta = 0)
Stream function: \( \psi = 0 \)
Vorticity:
\[ \psi(\xi, \eta = 1) = \psi(\xi, 0) + \psi_\eta(\xi, 0) \cdot 1 + \psi_{\eta \eta}(\xi, 0) \frac{1}{2} \] HOT

Using that \( \alpha(\xi, 0) = -\frac{x_y^2 + x_z^2}{\tau} \psi_\eta(\xi, 0) \)
We have:
\[ \alpha(\xi, 0) = -\frac{x_y^2 + x_z^2}{\tau} \left[ \psi(\xi, 0) - \psi(\xi, 1) \right] \]
Computational Fluid Dynamics I
Vorticity-Stream Function Formulation

Upper wall ($\eta = M$)
Stream function: $\psi = Q$

$$Q = \int_{-\infty}^{y} (udy - vdx) = \int_{\psi_u}^{\psi} d\psi = \psi - \psi_u$$

Vorticity:
$$\omega(\xi, M) = -2 \frac{\xi^2 + y^2}{J^2} [\psi(\xi, M) - \psi(\xi, M - 1)]$$

Inlet flow ($\xi = 0$)
Considering a fully-developed parabolic profile

$$u(0, y) = Cy(L - y)$$

$$Q = \int_{0}^{L} C(-y^2 + yL)dy = C \frac{L^2}{6}; \quad C = \frac{6Q}{L^2}$$

$$u(0, y) = \frac{6Q}{L}y(L - y), \quad \frac{\partial u}{\partial y} = \alpha$$

$$Q(y) = \frac{6Q}{L} \int_{0}^{L} (yL - y^2)dy = \frac{3}{2} \frac{Q}{L} \left( 2 \left( \frac{y}{L} \right)^3 - 2 \left( \frac{y}{L} \right) \right)$$

Outflow ($\xi = N$)
Typically, assuming straight streamlines

$$\frac{\partial \psi}{\partial n} = 0$$

If $\xi$ is normal to the outflow boundary, this yields

$$\frac{\partial \psi}{\partial \xi} = 0$$

If not, a proper transformation is needed for

$$\frac{\partial \psi}{\partial n}$$

Velocity-Pressure Formulation

The Navier-Stokes equations in primitive form

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{\partial p}{\partial x} + \alpha \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial p}{\partial y} + \alpha \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

and continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
Advection Terms

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{J} \left[ (u\frac{\partial y}{\partial x})_x + (v\frac{\partial x}{\partial y})_y \right] \]
\[ = \frac{1}{J} \left[ (y, u)_y - (\frac{\partial y}{\partial x})_x + (\frac{\partial x}{\partial y})_y + w_y \right] \]
\[ + (x, v)_x - (\frac{\partial x}{\partial y})_y + w_x \]
\[ = \frac{1}{J} \left[ \left( u(y, v)_y - v(x, v)_x \right) \right] \]
\[ = \frac{1}{J} \left[ \frac{\partial}{\partial x} (uU) + \frac{\partial}{\partial y} (uV) \right] \]

Contravariant Velocity

\[ U = u\gamma_0 - v\gamma_x; \quad V = v\gamma_x - u\gamma_y \]
\[ \begin{bmatrix} \gamma_x \\gamma_y \end{bmatrix} \quad \text{Unit tangent vector along } \xi = C \]
\[ \begin{bmatrix} \gamma_x \\gamma_y \end{bmatrix} \quad \text{Unit normal vector} \]
\[ U = (u, v) \cdot (\gamma_x, \gamma_y) = w_x - v_y \]

Therefore, \( U \) is in the \( \xi \) direction.
\( V \) is in the \( \eta \) direction.

Pressure Term

\[ \frac{\partial p}{\partial x} = \frac{1}{J} (p_x \gamma_x - p_y \gamma_y) \]

Diffusion Term

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{1}{J} \left( \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left( u \right) \right) \]
\[ = \frac{1}{J} \left( \frac{\partial (u_x, y_x)_x}{\partial x} \right) \]
\[ = \frac{1}{J} \left[ \left( u_x, y_x \right)_x + \frac{1}{2} (u_x, u_x)_x \right] \]

u-Momentum Equation

\[ \frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} = -\frac{1}{J} \left( \gamma_y \frac{\partial p}{\partial x} - \gamma_x \frac{\partial p}{\partial y} \right) \]
\[ + \frac{\partial}{\partial x} \left( qU - qU_0 \right) + \frac{\partial}{\partial y} \left( qV - qV_0 \right) \]

v-Momentum Equation

\[ \frac{\partial v}{\partial t} + \frac{\partial (uv)}{\partial x} + \frac{\partial (v^2)}{\partial y} = -\frac{1}{J} \left( \gamma_x \frac{\partial p}{\partial y} - \gamma_y \frac{\partial p}{\partial x} \right) \]
\[ + \frac{\partial}{\partial x} \left( qV - qV_0 \right) + \frac{\partial}{\partial y} \left( qV - qV_0 \right) \]
The momentum equations can be rearranged to

U-Momentum Equation

\[ \frac{\partial U}{\partial t} + \frac{\partial u_U}{\partial \xi} + \frac{\partial v_U}{\partial \eta} = -\frac{\partial p}{\partial \xi} + \alpha \left( \frac{\partial}{\partial \xi} \left( q_U - q_{\eta\xi} \right) + \frac{\partial}{\partial \eta} \left( q_U - q_{\eta\eta} \right) \right) - \xi \left( \frac{\partial}{\partial \xi} \left( q_{v\xi} - q_{\eta\xi} \right) + \frac{\partial}{\partial \eta} \left( q_{v\xi} - q_{\eta\eta} \right) \right) \]

V-Momentum Equation

\[ \frac{\partial V}{\partial t} + \frac{\partial u_V}{\partial \xi} + \frac{\partial v_V}{\partial \eta} = -\frac{\partial p}{\partial \eta} + \alpha \left( \frac{\partial}{\partial \xi} \left( q_V - q_{\eta\xi} \right) + \frac{\partial}{\partial \eta} \left( q_V - q_{\eta\eta} \right) \right) - \eta \left( \frac{\partial}{\partial \xi} \left( q_{v\xi} - q_{\eta\xi} \right) + \frac{\partial}{\partial \eta} \left( q_{v\xi} - q_{\eta\eta} \right) \right) \]

where

\[ u = \frac{1}{f} (Ux + Vy) \quad v = \frac{1}{f} (Vx + U\xi) \]

Stretched Grids

In the \( \xi - \eta \) plane, a staggered grid system can be used.

Same MAC grid and projection method can be used.

Special case:

\[ x = x(\xi) \quad a_1 = y_e^2 \]
\[ y = y(\eta) \quad a_2 = 0 \quad J = x_1 y_e \]

\[ u = \frac{1}{f} (Ux) = \frac{1}{x_1 y_e} (Ux) = \frac{U}{y_e} \]
\[ v = \frac{1}{f} (Vx) = \frac{1}{y_e x_1} (Vx) = \frac{V}{x_1} \]

\[ \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0 \]
\[ y_e \frac{\partial u}{\partial \xi} + \xi \frac{\partial v}{\partial \eta} = 0 \]
\[ \frac{1}{x_1} \frac{\partial u}{\partial \xi} + \frac{1}{y_e} \frac{\partial v}{\partial \eta} = 0 \]
Approximate the conservation equation \( \Delta \xi = \Delta \eta = 1 \)
\[
y_{i} = \frac{y_{i+1} - y_{i}}{1} = \Delta y_{i+1/2}
\]
\[
x_{i} = \frac{x_{i+1} - x_{i}}{1} = \Delta x_{i+1/2}
\]
\[
\frac{1}{\Delta x_{i}} \frac{\partial u}{\partial t} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial t} = 0 \hspace{1cm} \frac{1}{\Delta x_{i}} \frac{\partial u}{\partial x} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} = 0
\]

Define:
\[
\Delta y_{i} = \frac{1}{2} \left( \Delta y_{i+1/2} + \Delta y_{i-1/2} \right)
\]
\[
\Delta x_{i} = \frac{1}{2} \left( \Delta x_{i+1/2} + \Delta x_{i-1/2} \right)
\]

\[
\frac{1}{\Delta x_{i}} \frac{\partial u}{\partial t} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial t} + \frac{1}{\Delta x_{i}} \frac{\partial u}{\partial x} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} = 0
\]

\[
\frac{1}{\Delta x_{i}} \frac{\partial u}{\partial t} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial t} + \frac{1}{\Delta x_{i}} \frac{\partial u}{\partial x} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} = 0
\]

\[
\frac{1}{\Delta x_{i}} \frac{\partial u}{\partial t} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial t} + \frac{1}{\Delta x_{i}} \frac{\partial u}{\partial x} + \frac{1}{\Delta y_{j}} \frac{\partial v}{\partial y} \frac{\partial v}{\partial x} = 0
\]

The pressure equation is:
\[
\frac{
\begin{vmatrix}
\frac{P_{i+1/2,j+1} - P_{i-1/2,j+1}}{\Delta y_{j+1/2}} & \frac{u^{*}_{i+1,j+1} - u^{*}_{i-1,j+1}}{\Delta y_{j+1/2}} \\
\frac{v^{*}_{i+1,j+1} - v^{*}_{i-1,j+1}}{\Delta x_{i+1/2}} & \frac{P_{i+1/2,j} - P_{i-1/2,j}}{\Delta y_{j+1/2}}
\end{vmatrix}
\end{vmatrix}
\]

Which can be solved by iteration

For one dimensional stretched grids, we simply replace the global \( \Delta x \) by the local \( \Delta x \)
\[
\frac{u_{i+1/2,j} - P_{j}}{\Delta x_{i+1/2}} = \frac{u_{i+1/2,j} - P_{j}}{\Delta x_{i+1/2}}
\]
Therefore, numerical evaluation is preferred.

The use of exact evaluation introduces an additional

\[ x \approx f = \frac{\partial f}{\partial x} \nabla f + (\varepsilon) f \]

\[ \frac{\partial f}{\partial x} \Delta x + (\varepsilon) f \Delta x \]

Generalized Accuracy of Generalized Coordinates

\[ \text{Considered 1-D Stretching} \]

Taylor expansion

\[ f(x) = f(\xi) + \frac{\partial f}{\partial x} \frac{x}{\xi} \Delta x + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \left( \frac{x}{\xi} \right)^2 \Delta x^2 + \cdots \]

Further algebraic expansion gives

\[ f_1 - f_1 = f_1 + \frac{\Delta^2}{6} \left( f_1 + 3 f_{x+1} + \cdots \right) \Delta x^2 + \cdots \]

\[ \frac{\partial f}{\partial x} = f_1 + \frac{\Delta^2}{6} \left( f_1 + 3 f_{x+1} + \cdots \right) \Delta x^2 + \cdots \]

It may appear 2nd order accuracy, but…

If \( \xi \) is evaluated exactly

\[ \frac{\partial f}{\partial x} = f_1 + \frac{\Delta^2}{6} \left( f_1 + 3 f_{x+1} + \cdots \right) \Delta x^2 + \cdots \]

The use of exact evaluation of \( \xi \) introduces an additional (and dominant) term in the truncation error. Therefore, numerical evaluation is preferred.

By mapping a complex domain can be mapped into a rectangular one, where the techniques that we have covered can be used directly.

The governing equations, however, become more complex.

Although the geometry of the domain can be fairly complex, it must be possible to map it onto a rectangular domain. For more complex domains it is possible to use block-structured grids, but very complex shapes are more easily done by unstructured grids.
What about Veldman's results? Should I add?