Theory of Partial Differential Equations-I

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Examples of equations
\[
\begin{align*}
\frac{\partial f}{\partial t} + U \frac{\partial f}{\partial x} &= 0 \quad \text{Advection} \\
\frac{\partial f}{\partial t} - D \frac{\partial^2 f}{\partial x^2} &= 0 \quad \text{Diffusion} \\
\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} &= 0 \quad \text{Wave propagation} \\
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 0 \quad \text{Laplace equation}
\end{align*}
\]

Definitions
\begin{itemize}
\item The order of PDE is determined by the highest derivatives
\item Linear if no powers or products of the unknown functions or its partial derivatives are present.
\item Quasi-linear if it is true for the partial derivatives of highest order.
\end{itemize}

\[
\begin{align*}
\frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= f, \quad \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + 2xf &= 0 \\
\frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= f, \quad x^2 \frac{\partial f}{\partial x} + y^2 \frac{\partial f}{\partial y} &= f^2
\end{align*}
\]

Outline
\begin{itemize}
\item Basic Properties of PDE
\item Quasi-linear First Order Equations
  - Characteristics
  - Linear and Nonlinear Advection Equations
\item Quasi-linear Second Order Equations
  - Classification: hyperbolic, parabolic, elliptic
  - Domain of Dependence/Influence
\item Ill-Posed Problems
\item Conservative form
\item Navier-Stokes equations
\end{itemize}

Navier-Stokes equations
\[
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= - \frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= - \frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
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\]

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\end{align*}
\]
Consider the quasi-linear first order equation
\[ a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = c \]

where the coefficients are functions of \( x, y, \) and \( f, \) but not the derivatives of \( f: \)
\[ a = a(x,y,f) \]
\[ b = b(x,y,f) \]
\[ c = c(x,y,f) \]

The solution of this equations defines a single valued surface \( f(x,y) \) in three-dimensional space:

An arbitrary change in \( f \) is given by:
\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \]

The normal vector to the curve \( f=f(x,y) \)
\[ n = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \]

Same arguments in the y-direction. Thus \( n = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \)

The original equation and the conditions for a small change can be rewritten as:
\[ a \frac{\partial f}{\partial x} = c \Rightarrow \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) (a,b,c) = 0 \]
\[ df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \Rightarrow \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) (dx,dy,df) = 0 \]

Normal to the surface
Both \((a,b,c)\) and \((dx,dy,df)\) are in the surface!

Picking the displacement in the direction of \((a,b,c)\)
\[ (dx,dy,df) = dh(a,b,c) \]

Separating the components
\[ \frac{dx}{dh} = a; \quad \frac{dy}{dh} = b; \quad \frac{df}{dh} = c; \]
\[ \frac{dx}{dy} = \frac{a}{b} \]
The three equations specify lines in the x-y plane:
\[
\frac{dx}{ds} = a; \quad \frac{dy}{ds} = b; \quad \frac{df}{ds} = c;
\]

**Characteristics**

Given the initial conditions:
\[
x = x(s, t_0); \quad y = y(s, t_0); \quad f = f(s, t_0);
\]

the equations can be integrated in time:

slope \( \frac{dx}{dy} = \frac{a}{b} \)

Consider the linear advection equation
\[
\frac{\partial f}{\partial t} + U \frac{\partial f}{\partial x} = 0
\]

The characteristics are given by:
\[
\frac{dx}{dt} = 1; \quad \frac{ds}{dt} = U; \quad \frac{df}{ds} = 0;
\]

or
\[
\frac{dx}{ds} = U; \quad \frac{df}{ds} = 0;
\]

Which shows that the solution moves along straight characteristics without changing its value.

The solution is therefore:
\[
f(x, t) = g(x - Ut) \quad \text{where} \quad g(x) = f(x, t = 0)
\]

This can be verified by direct substitution:

Set \( \eta(x, t) = x - Ut \)

Then \( \frac{df}{dt} = \frac{df}{d\eta} \frac{d\eta}{dt} = \frac{df}{d\eta} (-U) \) and \( \frac{df}{dx} = \frac{df}{d\eta} \frac{d\eta}{dx} \frac{dx}{d\eta} = \frac{df}{d\eta} \frac{1}{d\eta} \)

Substitute into the original equation:
\[
\frac{df}{dt} + U \frac{df}{dx} = \frac{df}{d\eta} (-U) + U \frac{df}{d\eta} = 0
\]

Discontinuous initial data
\[
\frac{df}{dt} + U \frac{df}{dx} = 0 \quad \text{where} \quad f(x, t) = g(x - Ut)
\]

Since the solution propagates along characteristics completely independently of the solution at the next spatial point, there is no requirement that it is differentiable or even continuous.
Consider a nonlinear (quasi-linear) advection equation
\[ \frac{df}{dt} + U \frac{df}{dx} = 0 \]

The characteristics are given by:
\[ \frac{dt}{ds} = 1; \quad \frac{dx}{ds} = U; \quad \frac{df}{ds} = -f; \]

or
\[ \frac{dx}{dt} = U; \quad \frac{dt}{ds} = 1; \quad \frac{df}{ds} = 0; \]

\[ \Rightarrow f = f(0)e^{-t} \]

The slope of the characteristics depends on the value of \( f(x,t) \).

Moving wave with decaying amplitude

Why unphysical solutions?
- Because mathematical equation neglects some physical process (dissipation)

\[ \frac{df}{dt} + f \frac{df}{dx} + \varepsilon \frac{d^2 f}{dx^2} = 0 \]  
Burgers Equation

Additional condition is required to pick out the physically Relevant solution

Correct solution is expected from Burgers equation with \( \varepsilon \rightarrow 0 \)

Entropy Condition

Constructing physical solutions
\[ \frac{df}{dt} + f \frac{df}{dx} = 0 \]

In most cases the solution is not allowed to be multiple valued and the "physical solution" must be reconstructed using conservation of \( f \)

The discontinuous solution propagates with a shock speed that is different from the slope of the characteristics on either side
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- Navier-Stokes equations

**Quasi-linear Second order partial differential equations**

Consider

\[ a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = d \]

where

\[ a = a(x,y,f,f_x,f_y) \]
\[ b = b(x,y,f,f_x,f_y) \]
\[ c = c(x,y,f,f_x,f_y) \]
\[ d = d(x,y,f,f_x,f_y) \]

The second equation is obtained from

\[ \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \]

Thus

\[ \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = d \]

**Any high-order PDE can be rewritten as a system of first order equations!**

In matrix form

\[ \begin{pmatrix} \frac{\partial v}{\partial x} + b \frac{\partial v}{\partial y} + c \frac{\partial w}{\partial y} = d \\ \frac{\partial w}{\partial x} - \frac{\partial v}{\partial y} = 0 \end{pmatrix} \]

or

\[ u_x + Au_y = s \]

**Are there lines in the x-y plane, along which the solution is determined by an ordinary differential equation?**
The total derivative is
\[ \frac{dv}{dx} = \frac{dv}{dx} + \frac{\alpha}{y} \frac{dv}{dy} \]
where \( \alpha = \frac{dv}{dx} \)

Rate of change of \( v \) with \( x \), along the line \( y = y(x) \)

If there are lines (determined by \( a \)) where the solution is governed by ODE’s, then it must be possible to rewrite the equations in such a way that the result contains only \( \alpha \) and the total derivatives.

Add the original equations:
\[ l_1 \left( \frac{dv}{dx} + \frac{b}{a} \frac{dv}{dy} + c \frac{dv}{dy} \right) + l_2 \left( \frac{dv}{dx} - \frac{\alpha}{a} \frac{dv}{dy} \right) = l \frac{d}{dx} \]

Is this ever equal to
\[ l_1 \left( \frac{dv}{dx} + \alpha \frac{dv}{dy} \right) + l_2 \left( \frac{dv}{dx} - \alpha \frac{dv}{dy} \right) = l \frac{d}{dx} \]

For some \( l \)’s and \( \alpha \)

Characteristic lines exist if:
\[ l_1 \frac{b}{a} - l_2 = l_1 \alpha \]
\[ l_1 \frac{c}{a} = l_2 \alpha \]

Or, in matrix form:
\[
\begin{pmatrix}
\frac{b}{a} - \alpha & 0 \\
\frac{c}{a} & -\alpha
\end{pmatrix}
\begin{pmatrix}
l_1 \\
l_2
\end{pmatrix}
= 0
\]

The determinant is:
\[
\begin{vmatrix}
\frac{b}{a} - \alpha & -1 \\
\frac{c}{a} & -\alpha
\end{vmatrix}
= 0
\]

The equation has a solution only if the determinant is zero
\[
\begin{pmatrix}
\frac{b}{a} - \alpha & -1 \\
\frac{c}{a} & -\alpha
\end{pmatrix}
\begin{pmatrix}
l_1 \\
l_2
\end{pmatrix}
= 0
\]

The determinant is:
\[ A^T - \alpha I = 0 \]

Or, solving for \( \alpha \)
\[
\alpha = \frac{1}{2a} \left( b \pm \sqrt{b^2 - 4ac} \right)
\]
$\alpha = \frac{1}{2a} (b \pm \sqrt{b^2 - 4ac})$

- $b^2 - 4ac > 0$ Two real characteristics
- $b^2 - 4ac = 0$ One real characteristics
- $b^2 - 4ac < 0$ No real characteristics

Comparing with the standard form shows that:

- $a = 1$; $b = 0$; $c = -c^2$; $d = 0$;
- $b^2 - 4ac = 0^2 + 4 \cdot 1 \cdot c^2 = 4c^2 > 0$

Hyperbolic

$\alpha = \frac{1}{2a} (b \pm \sqrt{b^2 - 4ac})$

Comparing with the standard form shows that:

- $a = 1$; $b = 0$; $c = -1$; $d = 0$;
- $b^2 - 4ac = 0^2 - 4 \cdot 1 \cdot (-1) = -4 < 0$

Elliptic

$\frac{\partial f}{\partial x} = D \frac{\partial^2 f}{\partial y^2}$

Comparing with the standard form shows that:

- $a = 0$; $b = 0$; $c = -D$; $d = 0$;
- $b^2 - 4ac = 0^2 + 4 \cdot 0 \cdot D = 0$

Parabolic

$\frac{\partial^2 f}{\partial x^2} - c \frac{\partial^2 f}{\partial y^2} = 0$

Hyperbolic

$\frac{\partial f}{\partial x} = D \frac{\partial^2 f}{\partial y^2}$

Parabolic

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$\frac{\partial f}{\partial x} = D \frac{\partial^2 f}{\partial y^2}$

Parabolic
\[ \alpha = \frac{\partial y}{\partial x} \]

\[ \alpha = \frac{1}{2a} \left( b \pm \sqrt{b^2 - 4ac} \right) \]

\[ \frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \]

To find the characteristics

\[
\begin{align*}
\frac{\partial u}{\partial t} - c^2 \frac{\partial v}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} &= 0
\end{align*}
\]

\[ \Rightarrow \alpha = \pm c \]

To find the solution we need to find the eigenvectors

\[ l_1 \begin{bmatrix} \frac{\partial u}{\partial t} - c^2 \frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0 \end{bmatrix} \begin{bmatrix} -\alpha \\ -c^2 -\alpha \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = 0 \]

\[ + l_2 \begin{bmatrix} \frac{\partial u}{\partial t} - c^2 \frac{\partial v}{\partial x} = 0 \\ \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0 \end{bmatrix} \begin{bmatrix} -\alpha \\ -c^2 -\alpha \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = 0 \]

Take \( l_1 = 1 \)

For \( \alpha = +c \)

\[ -c \ l_1 - l_2 = 0 \]

\[ l_2 = -c \]

For \( \alpha = -c \)

\[ +c \ l_1 - l_2 = 0 \]

\[ l_2 = +c \]
\[
\frac{\partial u}{\partial t} - c^2 \frac{\partial v}{\partial x} = 0
\]
\[
\left( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} \right) - c \left( \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} \right) = 0
\]
Add the equations!

\[
\frac{\partial u}{\partial t} - c \frac{\partial v}{\partial t} - \frac{\partial u}{\partial x} = 0
\]
\[
\left( \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} \right) - c \left( \frac{\partial v}{\partial t} + c \frac{\partial v}{\partial x} \right) = 0
\]
Relation between the total derivative on the characteristic!

For \( \alpha = +c \):
\[
l_1 = 1 \quad l_2 = -c
\]
\[
\frac{du}{dt} - c \frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = +c
\]
\[
\frac{du}{dt} + c \frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -c
\]

For \( \alpha = -c \):
\[
l_1 = 1 \quad l_2 = +c
\]
\[
\frac{du}{dt} - c \frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = +c
\]
\[
\frac{du}{dt} + c \frac{dv}{dt} = 0 \quad \text{on} \quad \frac{dx}{dt} = -c
\]

Similarly: \( r_1 \) and \( r_2 \) are called the Riemann invariants.

The general solution can therefore be written as:
\[
f(x,t) = r_1(x - ct) + r_2(x + ct)
\]

where
\[
r_1(x) = \left. \frac{\partial f}{\partial x} - c \frac{\partial f}{\partial t} \right|_{x=0}
\]
\[
r_2(x) = \left. \frac{\partial f}{\partial x} + c \frac{\partial f}{\partial t} \right|_{x=0}
\]

Can also be verified by direct substitution.

Theory of Partial Differential Equations-III

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Ill-posed problems

Consider the initial value problem:
\[ \frac{\partial^2 f}{\partial t^2} = -\frac{\partial^2 f}{\partial x^2} \]
This is simply Laplace’s equation
\[ \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0 \]
which has a solution if \( \partial f/\partial x \) or \( f \) are given on the boundaries.

Here, however, this equation appeared as an initial value problem, where the only boundary conditions available are at \( t = 0 \). Since this is a second order equation we will need two conditions, which we may assume are that \( f \) and \( \partial f/\partial x \) are specified at \( t = 0 \).

The general solution can be written as:
\[ f(x,t) = \sum \limits_{k} a_k(t)e^{ikx} \]
where the \( a_k \)'s depend on the initial conditions.

Look for solutions of the type:
\[ f = a_k(t)e^{ikx} \]

Substitute into:
\[ \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} = 0 \]
to get:
\[ \frac{d^2a_k}{dt^2} = k^2a_k \]

Generally, both \( A \) and \( B \) are non-zero.

Long wave with short wave perturbations

\[ \frac{d^2a_k}{dt^2} = k^2a_k \]

General solution
\[ a_k(t) = Ae^{kt} + Be^{-kt} \]

\( A, B \) determined by initial conditions \( a(0), \ ika(0) \)

Generally, both \( A \) and \( B \) are non-zero

Therefore, \( a(t) \to \infty \) as \( t \to \infty \)

Ill-posed Problem
Ill-posed problems generally appear when the initial or boundary data and the equation type do not match. Frequently arise because small but important higher order effects have been neglected. Ill-posedness generally manifests itself in the exponential growth of small perturbations so that the solution does not "depend continuously on the initial data".

Why is the classification Important?
1. Initial and boundary conditions
2. Different physics
3. Different numerical method apply

The Navier-Stokes equations contain three equation types that have their own characteristic behavior. Depending on the governing parameters, one behavior can be dominant. The different equation types require different solution techniques. For inviscid compressible flows, only the hyperbolic part survives.

In the next several lectures we will discuss numerical solutions techniques for each class:
- Hyperbolic equations, including solutions of the Euler equations
- Parabolic equations
- Elliptic equations
Then we will consider advection/diffusion equations and the special considerations needed there. Finally we will return to the full Navier-Stokes equations.