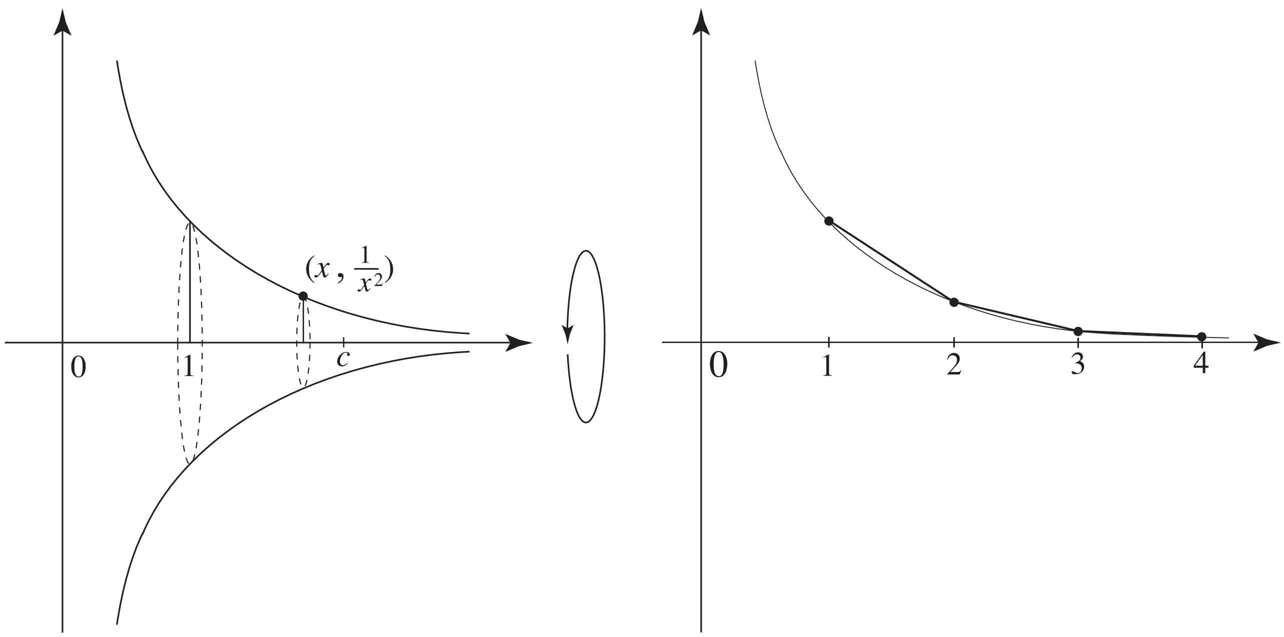
Quiz **Name**

Consider the function $f(x) = \frac{1}{x^2}$ with $x > 0$ and its graph and turn to Turn to Figure (a).

1. Show that the volume that the region under the graph of $f(x) = \frac{1}{x^2}$ over the interval $[1, +\infty)$ generates is $\lim_{c\to+\infty} \pi \int_1^c (\frac{1}{x^2})^2\, dx$. Show that the value of this improper integral is 1.



(a) (b)

2. Show that the surface area generated by the graph of $f(x) = \frac{1}{x^2}$ over the interval $[1, +\infty)$ is given by $\lim_{c\to+\infty} 2\pi \int_1^c \frac{1}{x^5}\sqrt{x^6+4}\, dx$. The antiderivative of $\frac{1}{x^5}\sqrt{x^6+4}$ that the application of the fundamental theorem requires seems impossible to come by and may not be an elementary function. (The site http://www.integral-calculator.com/# does not provide it.)

However, it is possible to show that show that $\lim_{c\to+\infty} 2\pi \int_1^c \frac{1}{x^5}\sqrt{x^6+4}\, dx$ is finite. The argument follows below. Provide the details.

3. Explain why $x^6 + 4 < (x + 2^{\frac{1}{3}})^6$. Conclude that $\sqrt{x^6+4} < (x + 2^{\frac{1}{3}})^3$, and show that

$$\int_1^c \frac{1}{x^5}\sqrt{x^6+4}\, dx < \int_1^c \big(x^{-2} + 2^{\frac{1}{3}}x^{-3} + 2^{\frac{2}{3}}x^{-4} + 4x^{-5}\big)\, dx = \big(-x^{-1} - \frac{1}{2^{\frac{2}{3}}}x^{-2} - \frac{2^{\frac{2}{3}}}{3}x^{-3} - x^{-4}\big)\Big|_1^c$$
$$= (1 + \frac{1}{2^{\frac{2}{3}}} + \frac{2^{\frac{2}{3}}}{3} + 1) - (\frac{1}{c} + \frac{1}{2^{\frac{2}{3}}}\frac{1}{c^2} + \frac{2^{\frac{2}{3}}}{3}\frac{1}{c^3} + \frac{1}{c^4}).$$

Show that therefore, $\lim_{c\to+\infty} 2\pi \int_1^c \frac{1}{x^5}\sqrt{x^6+4}\, dx < 2\pi(2 + \frac{1}{2^{\frac{2}{3}}} + \frac{2^{\frac{2}{3}}}{3}) < 20$. So the surface area of this region of infinite extent is finite. (The bound 20 can be improved by a more careful analysis to around 8.)