Additional Exercises for Chapter 1

More Greek Algebra from *The Greek Anthology* (Volume V of Book XIV).

29. I desire my two sons to receive the thousand staters of which I am possessed, but let the fifth part of the legitimate one's share exceed by ten the fourth part of what falls to the illegitimate one. [A stater was a Greek gold or silver coin. Its value varied from about \$120 to \$240 for the silver version and from about \$1200 to \$2400 for the gold.]

30. The Graces were carrying baskets of apples, and in each was the same number. The nine Muses met them and asked them for apples, and they gave the same number to each Muse, and the nine and the three had each of them the same number.

31. Mother, why dost thou pursue me with blows on account of the walnuts? Pretty girls divided them all among themselves. For Melission took two seventh of them from me, and Titane took the twelfth. Playful Astyche and Philinna have the sixth and third. Thetis seized and carried off twenty, and Thisbe twelve, and look there Glauce smiling sweetly with eleven in her hand. This one nut is all that is left to me.

32. The sun, the moon, and the planets of the revolving zodiac spun such a nativity for thee; for a sixth part of thy life to remain an orphan with thy dear mother, for an eight part to perform forced labor for thy enemies. For a third part the gods shall grant thee home-coming, and likewise a wife and a later-born son by her. Then thy son and wife shall perish by the spears of the Scythians, and then having shed tears for them thou shalt reach the end after twenty-seven years.

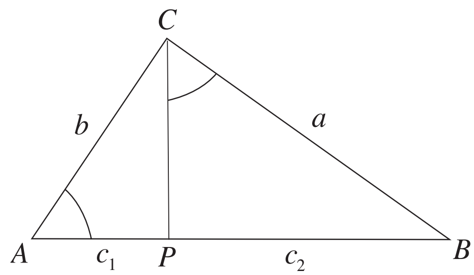
33. Of the four spouts one filled the whole tank in a day, the second in two days, the third in three days, and the fourth in four days. What time will all four take to fill it?

34. We three Loves stand here pouring out water for the bath, sending streams into the fair-flowing tank. I on the right, from my long-winged feet, fill it full in the sixth part of the day; I on the left, from my jar, fill it in four hours; and I in the middle, from my bow, in just half a day. Tell me in what sort of time we should fill it, pouring water from wings, bow and jar all at once.

Basic Geometry

35. Draw a random triangle and label its vertices A, B, and C. Let α, β and γ be the angles at A, B and C respectively. Extend the segment AB to a line in both directions. Place a point D on the same side of this line that C is on in such a way that the segment DB is parallel to AC. Use your figure to show that $\alpha + \beta + \gamma = \pi$. [This argument is straight out of Euclid's *Elements*.] Return to the triangle, pick any vertex, and draw a line through the vertex parallel to the side opposite the vertex. Now verify $\alpha + \beta + \gamma = \pi$ again.

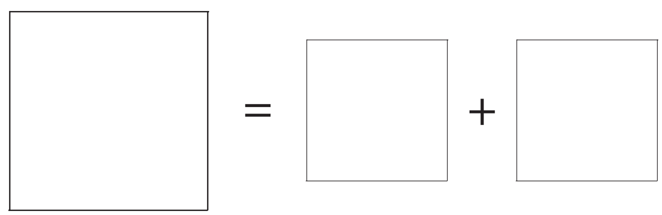
36. Let ΔABC be a right triangle and label the vertices so that AB is the hypothenuse. Let c be the length of the hypothenuse and let a and b be the lengths of the other two sides. Draw in the perpendicular segment PC from AB to C. See the figure below and note that $c = c_1 + c_2$. Why are the two triangles ΔPCA and ΔPCB both similar to ΔABC? Show that c is equal to both $\frac{a^2}{c_2}$ and $\frac{b^2}{c_1}$. Conclude that $a^2 + b^2 = c^2$. [There are many proofs of the Pythagorean



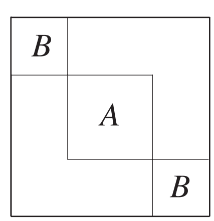
Theorem. This one has been attributed to the famous physicist Albert Einstein.]

More about $\sqrt{2}$

37. The proof that $\sqrt{2}$ is irrational is at least two thousand years old. In about the year 1990 the Princeton mathematician John Conway provided the following simple, but ingenious, geometric proof. Assume, if possible, that $\sqrt{2} = \frac{n}{m}$ where n and m are positive integers. So $n^2 = 2m^2$. Thus the area of a square of side length n is equal to the sum of the areas of two



squares of side length m. Let n be the smallest positive integer for which this holds. In other words, n is the smallest positive integer so that a square of side length n has an area that is the sum of the areas of two identical squares with side length a positive integer. How do you know that - under the given the assumption - there is such a smallest n? Now slide the two smaller squares into the larger one as shown below. Show that the square A and the two



identical squares B have integer sides and that the area of A is twice the area of B. Why can Conway now conclude that $\sqrt{2}$ must be irrational?

One of my presents for Christmas 2003 was the book *Prime Obsession* by John Derbyshire, both mathematician and writer. It is a wonderful and sophisticated account about prime numbers and the mathematicians for whom they became an object of intense pursuit. In one of the notes at the end of the book, the author provides a proof of the irrationality of $\sqrt{2}$ that he attributed to a T. Estermann. When I realized that this proof was the algebraic version of the proof that Conway claimed to be new in the 1990's, I dropped an e-mail to John Derbyshire with a request for the precise reference to the literature. On January 15, 2004, he sent this reply:

"I was a student of Teddy Estermann's at University College, London, and got that proof from the horse's mouth. The only book of his that I own is *Complex Numbers and Functions*, which hasn't got it. However, the LMS obituary

http://www.numbertheory.org/obituaries/LMS/estermann/index.html

has a mention at the end, with the proof and a reference:

The irrationality of $\sqrt{2}$, *Math. Gaz.* (408) 59 (1975), 100.

Teddy, though somewhat decayed when I knew him, was wonderful fun. Dry and noncommital in class, he was a fund of stories and jokes if you met him outside. (I met him several times at concerts - we were both great concert-goers.) He hated Hardy [a famous British mathematician] ... or at any rate, Hardy's book *Pure Mathematics*. The wonder of his classroom performances was his rendering (on the chalk-board, I mean) of the letter x, a rococo construction that managed to include within itself the infinity sign. All breathing stopped in the class when Teddy wrote an x."

Best, John Derbyshire

38. Recover Teddy Estermann's proof by translating Conway's geometric argument into algebra. [Hint: Keep the assumption that there are positive integers n and m such that $n^2 = 2m^2$ and that n is the smallest such. Refer back to Conway's proof and express the equation area A = 2(area B) in algebraic form.]

The Golden Section

In Book VI of his *Elements*, Euclid tells us that "a straight line is cut in 'extreme and mean ratio' when the whole line is to the greater segment as the greater is to the less." Let's analyze what he says algebraically. Let S be his line segment and let the point C divide S into two pieces of lengths a and b with $a \geq b$. Note that the segment has length $a+b$. So C cuts the segment S in 'extreme and mean ratio' if $a+b$ is to a as a is to b, in other words, if $\frac{a+b}{a} = \frac{a}{b}$. Observe that if this is so, then

$$\frac{a}{b} = \frac{b}{a} + 1 = \frac{1}{\frac{a}{b}} + 1 = \left(\frac{a}{b}\right)^{-1} + 1.$$

So the ratio $\frac{a}{b}$ satisfies the equation $x - x^{-1} - 1 = 0$ or $x^2 - x - 1 = 0$.

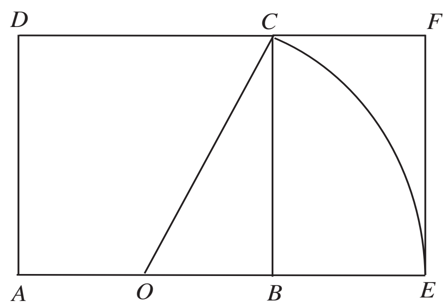
39. Suppose that C cuts the segment S in extreme and mean ratio.

i. Show that $\frac{a}{b} = \frac{1+\sqrt{5}}{2}$ and that this is an irrational number. [Hint: Apply Exercise 12 of Chapter 1 to show that $\sqrt{5}$ is irrational.]

ii. Check that $\frac{1+\sqrt{5}}{2} \approx 1.618$.

In today's terminology, such a cut at C is the *golden cut* or *golden section* of the segment S, and $\frac{a}{b} = \frac{1+\sqrt{5}}{2}$ is the *golden ratio.* If the lengths of the sides of a rectangle are in golden ratio to each other, then the rectangle is a *golden rectangle.*

40. Construct a rectangle in the following way. See the figure below. Begin with a square $ABCD$. Let O be the midpoint of AB and draw a circular arc centered at O from the vertex C to the



point E on the extension of the segment AB. Complete DAE to the rectangle $DAEF$. Verify that the rectangles $DAEF$ and $CBEF$ are both golden rectangles.

The golden ratio, also known as the *divine proportion*, is thought (without any supporting evidence) to have had great aesthetic appeal in Antiquity and that it was used by the Greeks as a principle of proportion in architecture, art, and music. Even today it is often asserted that the facade of the Parthenon on the Acropolis in Athens, built in the fifth century B.C. and one of the world's most famous structures, was designed to fit tightly into a golden rectangle.

More about Rational and Irrational Numbers

41. Express the rational number $1.77777\ldots$ as a fraction of two integers.

42. The number π is equal to $3.141592654\ldots$. Find integers n and m such that the rational number $3\frac{n}{m}$ approximates π with four decimal place accuracy.

43. An important irrational number e (we will encounter it in Chapter 10) has the decimal expansion $2.718281828\ldots$. Find integers n and m both less than 100,000 such that the rational number $2\frac{n}{m}$ approximates e with nine decimal place accuracy.

44. Can the rectangles below be subdivided into a finite array of non-overlapping identical squares, possibly very tiny?

i. The rectangle with sides 2 and 3? In many different ways? What about the rectangles with sides 5 and 7?

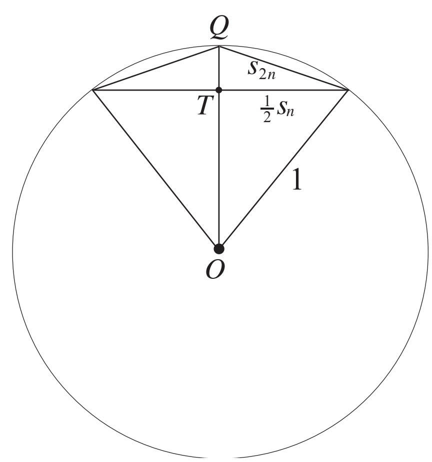
ii. The rectangle with sides $\sqrt{2}$ and $\sqrt{8}$?

45. Consider a rectangle R and let the lengths of its sides be a and b respectively. Show that if R can be subdivided into a finite array of non-overlapping identical squares, then the ratio $\frac{a}{b}$ must be a rational number. [Hint: What is the connection between a, b, and the length s of one of the identical squares?] Show next that if the ratio $\frac{a}{b}$ is a rational number, then R can be so subdivided.

46. Which of the following rectangles can be subdivided into identical non-overlapping squares: The rectangle with sides 1 and $\sqrt{2}$? A golden rectangle? The rectangle with sides $\sqrt{2}$ and $\sqrt{3}$? [Hint: Use the Fundamental Theorem of Arithmetic from Exercise 12.] The rectangle with sides $\sqrt{45}$ and $\sqrt{20}$?

Approximating π

Take a circle of radius 1 and let O be its center. Let n be some positive integer and place n points on the circle in such a way that any two consecutive points are the same distance apart. The figure obtained by connecting consecutive points with line segments is called a regular polygon with n sides, more compactly called a regular n-gon. Denote the length of any one of its sides (they are all equal) by s_n. Placing another point equidistant between each two consecutive points of this



polygon gives us a total of $2n$ points. Connecting every two consecutive points of this larger set of points gives us a regular $2n$-gon. Denote the length of one of its sides by s_{2n}. The figure below shows one side of the n-gon and two consecutive sides of the $2n$-gon.

47. Show that

i. $OT^2 = 1 - \frac{s_n^2}{4}$.

ii. $QT^2 = \left(1 - \frac{\sqrt{4-s_n^2}}{2}\right)^2$.

iii. $s_{2n}^2 = 2 - \sqrt{4 - s_n^2}$.

48. Show that $s_6 = 1$. Then use Exercise 47 to conclude that $s_{12} = \sqrt{2-\sqrt{3}}$, $s_{24} = \sqrt{2-\sqrt{2+\sqrt{3}}}$, $s_{48} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, and $s_{96} = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$. Use this last equation to show that $\pi = 48s_{96} \approx 3.14103 \approx \frac{22}{7}$. Recall that $\pi \approx \frac{22}{7}$ was an approximation obtained by Archimedes.

More about Circular Arcs

49. An arc on a circle is 4 inches long. If the radius of the circle is 5 inches, find (in radians) the angle determined by the endpoints of the arc and the center of the circle. If the angle is 21°, find the radius of the circle.

50. The hour hand of the clock on a clocktower is 2 feet long and the minute hand is 2.5 feet long. The clock loses one minute per hour. How many feet does the tip of the hour hand travel in 72 hours? How many feet does the tip of the minute hand travel during this time?

51. Take the Earth to be a sphere of radius $r_E = 3950$ miles. Given that the Earth rotates about its north-south axis once every 24 hours, estimate how fast a point P on the equator is moving around the Earth's axis. How fast is a point P on one of the poles moving? What can you say about any other point P on the surface of the Earth? [Hint for the last question: Let Q be the point on the equator due south (or north) of P. Let C be the center of the Earth and let α be the angle between the radii CP and CQ. Sketch a figure that shows the Earth as well as the circle traced out by P. Show that this circle has circumference $2\pi r_E \cos\alpha$.]

Understanding Archimedes's Number Scheme

Take any positive integer n and consider the sequence

$$n, n^2, \ldots, n^n, n^{n+1}, \ldots, n^{2n}, n^{2n+1}, \ldots, n^{3n}, \ldots, n^{n\cdot n} = n^{n^2} = (n^n)^n$$

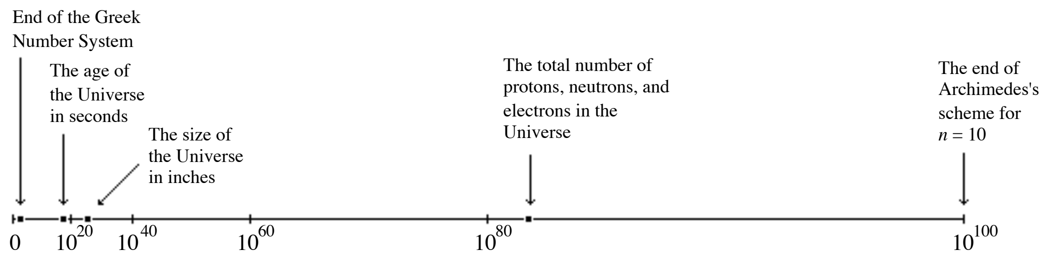
of consecutive powers of n. Since the exponents provide a count, we see that there are n^2 numbers in the sequence. For $n = 2$, the sequence is 2, 4, 8, and $2^4 = 16$. Observe that the gaps between the numbers are 2, 4, and 8, and that they grow by a factor of 2 from step to step. For $n = 3$, the sequence is

$$3,\ 9,\ 27,\ 81,\ \cdots,\ 3^9 = 19{,}683.$$

Now the gaps are $6, 18, 54, \ldots$. They grow by a factor of 3 from step to step. In the general case, the gaps grow by the factor n from step to step.

52. What is the last term of the sequence of Archimedes for $n = 4$? For $n = 5$, and $n = 6$?

For $n = 10$, there are $10^2 = 100$ numbers in the sequence with the last one equal to $10^{10\cdot 10} = 10^{100}$. This number is larger than the number of stable elementary particles in the entire universe. Physicists have estimated that the combined number of protons, neutrons, electrons, photons and neutrinos is no greater than 10^{90}.



Observe that the diagram is not drawn to scale

The number scheme that the *Sandreckoner* discusses is much larger still. Archimedes takes n to be the very largest number that could be written down in Greek notation plus 1. In other words, he takes $n = 100{,}000{,}000 = 10^8$. So the first few numbers that Archimedes considers are

$$10^8,\ 10^{16},\ 10^{24},\ 10^{32},\ 10^{40},\ 10^{48}, \ldots .$$

The twelfth number is $10^{8\cdot 12} = 10^{96}$. There are $(10^8)^2 = 10^{16}$ numbers in all, and the last one is the enormous number $10^{8\cdot 10^{16}}$.

Applying the Method of Stellar Parallax

53. The muse of astronomy Urania sees a Greek column off in the distance. Behind it on the horizon she recognizes the familiar outline of Mount Olympus and notices that the top of the column is in line with a certain distinctive feature of the mountain. Urania moves twelve strides to the left and observes a shift in the position of the top of column against the backdrop of the mountain. It is now aligned with a second distinctive feature. Returning to the original spot she estimates the angle between her lines of sight to the two features of the mountain to be about 5 degrees. After concentrated thought, Urania concludes that the column is about 140 strides away. How did she reach this conclusion?

54. Team Project: Several students go to a "spot" from which they have a clear view of a building that is a few hundred yards (say 200 yards) away. It should have a long facade that is roughly perpendicular to the line of sight. A student $*$ is selected and takes a position a number of paces, say twenty, from the spot (in the direction of the building). The "pace" is the operative unit of length. The other students mark out a "base" line (a few paces long) through the spot and roughly parallel to the facade of the building. The students then use the method of stellar parallax to estimate the number of paces from the base line to $*$. Finally, they pace off the distance to $*$ and compare it with the predicted value.

Properties of Similar Triangles

55. How would you and your two roommates go about estimating the height of the Hesburgh Library with a 10 foot tape measure? Discuss.

Essay Questions

A. Being able to measure time is important in lots of different contexts ranging from athletic events to cooking. What is the connection between the measurement of time and the real numbers?

B. Discuss the accuracy of Eratosthenes's estimate of the radius of the Earth. Where are some potential problems?

C. What would you say in response to the assertion: What Aristarchus did is well and good, but his estimates are so far from the actual values that what he did is essentially useless.

D. What do you think about Archimedes's number scheme? A useful speculation?