Additional Exercises for Chapter 2

All angles in these exercises are understood to be given in radians. We begin with problems related to Proposition 2.1 of the text. In reference to this proposition - see Corrections and Typos for Chapter 2 - note that the point $P$ must be outside the arc in question. Let $A$ and $B$ be two points on a circle. The two points determine two arcs. Moving in the same direction around the circle, there is the arc from $A$ to $B$ and then there is the arc from $B$ to $A$. When one of the two arcs is under consideration, the other arc will be referred to as the complementary arc.

Angles, Arcs, and Circles

14. Consider any circle and any arc on it. Let $A$ and $B$ be the endpoints of the arc and let $\theta$ be the angle that the arc and the center of the circle determine. See the figure below. Show that $\angle APB = \pi - \frac{\theta}{2}$ for any point $P$ on the arc except for the points $A$ and $B$. [Hint: Apply Proposition 2.1 to $P$ and the complementary arc.]

15. Consider a triangle inscribed in a circle. Show that if a side of the triangle is a diameter of the circle, then the angle opposite to that side is a right angle. Also, if one of its angles is a right angle, then the side opposite that angle is a diameter.

16. You are a very smart ant. You know geometry and you can measure distances and angles with complete precision. You have behaved badly and the authorities of your colony have confined your movement to a circular arc. After you have paced back and forth along the arc of your flat prison for a while, the thought occurs to you that it should be possible to compute the radius of the circle on which the arc lies. Suddenly there is the spark of an idea and you realize that you have solved the problem. What is your solution?

17. You, the ant of the previous exercise, have behaved well and the conditions of your confinement are improved. You can now move freely anywhere within the circle on which your prison lies. Always eager to exercise your mind, you decide to locate with accuracy the center of this circle. After some thought, you know what to do. How will you proceed?

18. Let $A$ and $B$ be two distinct points. Let $P$ be the midpoint of the segment $AB$. Let $L$ be the
line through $P$ that is perpendicular to the segment $AB$. This $L$ is called the perpendicular bisector of the segment $AB$. Show that the points on $L$ are precisely the points that are equidistant from $A$ and $B$.

19. Let $A$, $B$, and $C$ be points in the plane that do not all fall on the same line. Show that there is a circle through $A$, $B$, and $C$. It follows that any triangle can be inscribed in a circle. [Hint: Let the point $O$ be the intersection of the perpendicular bisectors of the sides $AB$ and $BC$. Why is $O$ the center of the circle in question? Use Exercise 18.]

20. Consider a circle and any arc on it. Let $A$ and $B$ be the endpoints of the arc. Let $C$ be any point on the circle but outside the arc and let $\angle ACB = \alpha$. Consider the line determined by $A$ and $B$ and suppose that $P$ is any point in the plane on the same side of the line as $C$. Show that if $\angle APB < \alpha$ then $P$ is outside the circle and if $\angle APB > \alpha$ then $P$ is within the circle. So $P$ is on the circle precisely $\angle APB = \alpha$. [Hint: Let $P$ be outside the circle. Move from $P$ within the lines $PA$ and $PB$ to a point $P'$ on the circle and note that $\angle APB < \angle AP'B = \alpha$.]

**More Trig and Triangles**

In several of the problems below, the arguments for acute angles and triangles and obtuse angles and triangles are a little different. Recall that an angle $\theta$ is acute if $0 \leq \theta \leq \frac{\pi}{2}$ and it is obtuse if $\frac{\pi}{2} < \theta \leq \pi$. A triangle is acute if all of its three angles are acute, and obtuse if one of its angles is obtuse. Section 1.4 of the text defined $\sin \theta$ and $\cos \theta$ for any acute $\theta$. Now let $\theta$ be obtuse. Then $0 \leq \pi - \theta \leq \frac{\pi}{2}$ and we define $\sin \theta = \sin(\pi - \theta)$ and $\cos \theta = -\cos(\pi - \theta)$. The definitions of the trigonometric functions will be taken up again in Section 4.4.

21. Let $a$, $b$, and $c$ be any three positive numbers. Is there a triangle that has $a$, $b$, and $c$ as the lengths of its sides?

22. Consider any triangle. Let $a$, $b$ and $c$ be the lengths of its sides as shown, and let $\theta$ be the angle opposite the side of length $c$. Verify that $c^2 = a^2 + b^2 - 2ab \cos \theta$. This equation is known
as the Law of Cosines. Deduce that if \( c^2 = a^2 + b^2 \) then \( \theta = 90^\circ \) and the triangle is a right triangle. [Hint: Place the triangle \( \triangle ABC \) into a circle and complete it to a quadrilateral by drawing \( AD \) parallel to \( CB \). Draw the perpendiculars from \( A \) and \( D \) to \( CB \), and apply Ptolemy’s Theorem to the quadrilateral \( ACBD \). See the figure above. The figure and the hint consider the situation where \( \theta \) is acute. Provide the argument for the case when \( \theta \) is obtuse.]

23. Two sides of a triangle have lengths 7 and 11. The angle between these two sides is \( \frac{\pi}{5} \). What is the length of the third side?

24. Consider any arc on a circle of radius \( r \) and let \( A \) and \( B \) be its end points. Let \( P \) be any point on the circle but not on the arc and let \( \angle APB = \alpha \). Verify that \( \text{arc} \ AB = 2r\alpha \) and \( AB = 2r(\sin \alpha) \). [Hint: First do the case where \( \alpha \) is acute. Apply Propositions 2.1 and Exercise 15.]

25. A triangle has sides \( a, b, \) and \( c \). With \( b \) as base, its height is \( h \). Show that the radius \( r \) of the circle on which the three vertices of the triangle lie is \( r = \frac{ac}{2h} \). [Hint: Compute the sine of one of the angles of the triangle in two ways.]

26. A triangle has sides of lengths 7 and 11 and (with the third side as base) height 4. Show that the radius \( r \) of the circle on which the three vertices of the triangle lie is \( r = \frac{77}{8} \).

27. Consider an \( n \)-sided polygon inscribed in a circle. Assume that it is regular, in other words, that all its vertices lie on the circle and that all of its sides have the same length. Let \( r \) be the radius of the circle and let \( s \) be the length of a side of the polygon. Show that \( s = 2r \sin \frac{180^\circ}{n} \). [Hint: Apply Proposition 2.1.]

28. A regular pentagon is inscribed in a circle of radius 1. What is the length of one of its sides?

29. Consider any triangle and let its sides have lengths \( a, b, \) and \( c \) respectively. Let \( \alpha \) be the angle opposite the side with length \( a \), \( \beta \) the angle opposite the side with length \( b \), and \( \gamma \) the angle opposite the side of length \( c \). Show that

\[
\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c},
\]

where all angles are given in radians. This relationship is known as the Law of Sines.

30. You are given a triangle \( \triangle ABC \). The length of the side \( AB \) is 8 and the angles at \( A \) and \( B \) are
\( \frac{\pi}{5} \) and \( \frac{\pi}{7} \) respectively. Determine the lengths of the other two sides of the triangle.

**Highways and Billboards**

A car moves along a straight stretch of highway. A passenger \( P \) looks out at an approaching billboard. Concentrate on the angle \( \angle APB \) between the passenger’s lines of sight to the left and right edges of the billboard. A look at the diagram below confirms that \( \angle APB \) first increases, but then decreases again. Therefore at some point before the car passes the billboard \( \angle APB \) attains a maximum value. Why is this so? Because the billboard is not parallel to the highway (this is so for a typical billboard), observe that a similar phenomenon occurs (with the backside of the billboard) as the car moves away. The question is: Can the two points at which \( \angle APB \) attains these maximum values be determined in some way? We begin the answer with some geometry.

**31.** Consider a line \( L \) and a segment \( AB \) in the plane that does not intersect the line and is not parallel to it. Show that there are two circles with the following property: The points \( A \) and \( B \) are both on the circle and the line \( L \) is a tangent of the circle. [Hint: To see this, sketch the following figure. Draw the perpendicular bisector of \( AB \). For any point \( C \) on this bisector let \( CP \) be the perpendicular to \( L \). Now proceed as follows. Start with a point \( C \) such that
CP < CB. Slide the perpendicular segment CP in the direction of AB and continue until CP > CB. It follows that somewhere along the way there must have been some point C on the perpendicular bisector of AB such that CP = CB. Why is this point C the center of one of the circles that you were asked to find? Refer to the figure on the next page. How will you go about locating the second circle?

The answer to the billboard question is provided by the next exercise.

32. Consider a line L and segment AB in the plane. Assume that the segment does not intersect L and is not parallel to L. Let P be any point on L and consider the angle \( \varphi = \angle APB \). Consider the two circles that have the points A and B on them and L as tangent. Show that \( \varphi \) attains its maximum values at the two points on L that are tangent to the circles. What can you say about these two maximal values of \( \varphi \)? Which of the two is larger? [Hint: By Exercise 31 such circles exist. To finish the problem apply Exercises 20.]

33. Suppose that the billboard is parallel to the highway. What is the answer to the angle problem in this case?

Similar Triangles

Consider triangles T and T'. Recall that T and T' are similar, if there is a correspondence between the vertices of T and T' so that the angles at corresponding vertices are equal. Notice, that if two angles of T are respectively equal to two angles of T', then the remaining two angles must be equal as well and T' and T' are similar. Why?

34. Assume that the four vertices of a quadrilateral fall on a circle. The four sides and the two diagonals determine four triangles. Consider a pair of these triangles that are not adjacent and show that they are similar.

35. Consider two triangles. Show that if the triangles are similar, then there is a correspondence \( a \rightarrow a', \ b \rightarrow b', \text{and} \ c \rightarrow c' \) between their sides and a constant d such that

\[
a' = da, \quad b' = db, \quad \text{and} \quad c' = dc.
\]

Is it true, conversely, that if there is there is a correspondence between the sides of the triangles and a constant so that the above equalities hold, then the two triangles are similar? [Hint: Use the Law of Sines.]

Revisiting some Trig Formulas

36. Let \( \alpha \) and \( \beta \) be acute angles. Consider a circle with diameter equal to 1 and and let CD be a diameter. Choose points A and B on the circle such that \( \angle ACD = \alpha \) and \( \angle BCD = \beta \). Apply Ptolemy’s Theorem to verify that \( \sin(\alpha + \beta) = (\sin \alpha)(\cos \beta) + (\cos \alpha)(\sin \beta) \).
[Hint: Use the figure below and Exercises 15 and 24.]

We continue with pictorial verifications of some basic trigonometric formulas. The angles \( \alpha \) and \( \beta \) of Exercises 37, 38, and 39 need to satisfy, \( 0 < \alpha + \beta < \frac{\pi}{2} \). The solutions make use of the two figures that follow below. In each of the two figures, the large right triangles with angles \( \alpha \) and \( \beta \) are constructed first. They are then completed by the surrounding rectangles.

37. Check that the lengths of the segments in the figure above are as indicated there and then verify the formulas

\[
\sin(\alpha + \beta) = (\sin \alpha)(\cos \beta) + (\cos \alpha)(\sin \beta) \quad \text{and} \\
\cos(\alpha + \beta) = (\cos \alpha)(\cos \beta) - (\sin \alpha)(\sin \beta).
\]
38. Use the identities from the exercise above to verify the "half-angle" formulas
   i. \( \sin 2\alpha = 2(\sin \alpha)(\cos \alpha) \),
   ii. \( \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha \), and
   iii. \( \cos 2\alpha = 1 - 2\sin^2 \alpha \).

39. Check that the lengths of the segments in the figure below are as indicated and then verify

the formula

\[
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - (\tan \alpha)(\tan \beta)}.
\]

Some Inverse Trigonometry

40. Let \( \alpha \) and \( \beta \) be acute angles such that \( \tan \alpha = \frac{1}{2} \) and \( \tan \beta = \frac{1}{3} \). Use the diagram below (the one on the left) to verify that \( \alpha + \beta = \frac{\pi}{4} \).
41. Let $\alpha$ and $\beta$ be acute angles such that $\tan \alpha = 1$ and $\tan \beta = 2$ and place them into the diagram on the previous page (the one on the right) as shown. Show that the angle $\gamma$ in the diagram satisfies $\tan \gamma = 3$. Use the diagram to verify that $\alpha + \beta + \gamma = \pi$.

**Note:** The last five exercises were taken from the volume R. B. Nelson, *Proofs without Words*, The American Mathematical Association of America, 1993.