Additional Exercises for Chapter 5

About Lines, Slopes, and Tangent Lines

39. Find an equation for the line through the two points \((-2, 7)\) and \((5, 2)\).

40. What is the slope-intercept form of the equation of the line given by \(3x + 5y + 2 = 0\)?

41. Find an equation of the line with slope \(-3\) through \((-6, -7)\).

42. Sketch the line with slope \(-2\) and \(y\)-intercept \(-1\).

43. Two lines \(L_1\) and \(L_2\), neither of them vertical nor horizontal, are given. Let their slopes be \(m_1\) and \(m_2\) respectively. Show that if the lines are perpendicular, then \(m_2 = -\frac{1}{m_1}\). Conversely, show that if \(m_2 = -\frac{1}{m_1}\), then the lines are perpendicular. [Hint: Move the lines without changing their slopes so that they both go through the origin \(O\). Let the points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\) and the angles \(\alpha, \alpha', \beta, \) and \(\beta'\) be as shown in the figure. If \(\angle P_1OP_2 = 90^\circ\), then the two triangles depicted are similar. Why? So \(m_2 = -\frac{1}{m_1}\). Conversely, ... ]

44. Sketch the graph of the equation \(y^2 = 2x + 7\). Let \(P = (x, y)\) be any point on this graph with \(y \neq 0\), and use Leibniz’s method to compute the slope of the tangent to the curve at that point. Find the slope of the tangent at the point \((1, -3)\) and determine an equation of this tangent line.

45. Consider the parabola \(y = x^2\) and the line \(y = 3x - 4\). Show that they do not intersect. Move the line toward the parabola without changing its slope. At what point will it first touch the parabola?
Ans: \((\frac{3}{2}, \frac{9}{4})\)

46. Consider the circle \(x^2 + y^2 = r^2\). Let \((x_0, y_0)\) be any point on the circle with \(y_0 \neq 0\) and use Leibniz’s tangent method to show that the slope of the tangent line to the circle at \((x_0, y_0)\) is equal to \(-\frac{x_0}{y_0}\).

i. Consider the radius of the circle from the origin to the point \((x_0, y_0)\) and show that it is perpendicular to this tangent.

ii. Suppose that \(r = 1\). Consider a line \(y = -\frac{1}{3}x + b\) with slope \(-\frac{1}{3}\) and \(y\)-intercept \(b\). For which constants \(b\) is the line tangent to the circle?

Ans: \(b = \pm \frac{1}{3}(\sqrt{10})\)

47. Use the Leibniz tangent method for \(y^3 = 3x^2 + 7\) to show that the slope of the tangent at a point \(P = (x, y)\) on the graph of this equation is equal to \(m_P = \frac{2x}{(3x^2+7)^{\frac{1}{3}}}\).

Use Facts (not limits) to Compute Derivatives and Antiderivatives

48. Compute the derivatives of the functions \(f(x) = 2x - 3x^2\), \(g(x) = 4x^{\frac{1}{3}} + 5x^{-1}\), \(h(x) = 6x^3 - 7x^{\frac{1}{3}}\).

49. Use the conclusion of Exercises 46 and 47 to show that the derivative of \(f(x) = \sqrt{1-x^2}\) is \(f'(x) = \frac{-x}{\sqrt{1-x^2}}\) and that the derivative of \(f(x) = (3x^2 + 7)^{\frac{1}{3}}\) is \(f'(x) = \frac{2x}{(3x^2+7)^{\frac{1}{3}}}\).

50. Consider the function \(f(x) = 5x^2 - 2x^3\) and let \(P = (x, y)\) be some point on its graph. Determine an equation for the tangent line to the graph at \(P\). [Hint: Change notation.]

51. Find antiderivatives for the functions \(f(x) = 1 - 3x^3 + 2x^{-\frac{1}{2}}\), \(g(x) = -\frac{1}{3}x^{-2} + 8x^{\frac{1}{3}}\), and \(h(x) = -4 + 3x^{-2} + 7x^{\frac{1}{3}}\).

52. Use the results of Exercise 49 to find antiderivatives for \(g(x) = \frac{x}{\sqrt{1-x^2}}\) and \(g(x) = \frac{x}{(3x^2+7)^{\frac{1}{3}}}\).

53. Let \(P = (x, y)\) be a point on the graph of some function \(f\). Illustrate the meaning of the ratio \(\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}\). What do you think that \(\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}\) is equal to? Then prove that your answer is correct.

A Problem from Tokyo

54. Water is poured at a constant rate into each of the three drinking glasses and three vases shown below. Each vessel is empty at the start. The graph under the glass labeled (a) represents the height of the water in the glass as a function of time. Think carefully what is going on in the other five cases and draw the time/height graphs for each. [This problem was adapted from a 2006 advertisement in a Tokyo subway for a tutorial service for middle school students.]
More Areas and Differentials

55. Consider the function \( g(x) = 9 - x^2 \) with \( 0 \leq x \leq 1 \). Select the points

\[
0 \leq 0.3 \leq 0.5 \leq 0.8 \leq 1
\]
on the \( x \)-axis between 0 and 1 and compute the sum of the areas \( y \cdot dx = g(x) \cdot dx \) of all the rectangles that this set of points determines. Do so with three decimal place accuracy. This
sum is an approximation of the area under the graph of \( g(x) = 9 - x^2 \) over \( 0 \leq x \leq 1 \). Repeat this computation (again with three decimal accuracy) with the points

\[
0 \leq 0.2 \leq 0.3 \leq 0.5 \leq 0.7 \leq 0.8 \leq 1
\]

to get another approximation of the area under the graph. Which of these two approximations would you expect to be better? Use Archimedes’s Theorem to get the exact answer.

Ans: 8.779, 8.751, \( \frac{26}{3} \approx 8.667 \).

56. Consider the function \( f(x) = 16 - x^2 \) with \(-2 \leq x \leq 2\). Select the points

\[
-2 \leq -1.6 \leq -1 \leq -0.4 \leq 0 \leq 0.5 \leq 1 \leq 1.4 \leq 2
\]
on the \( x \)-axis between -2 and 2. Compute the sum of the areas \( y \cdot dx = f(x) \cdot dx \) of all the rectangles that this setup determines. Do so with three decimal place accuracy. This sum is an approximation of the area under the graph of \( f(x) = 16 - x^2 \) over \(-2 \leq x \leq 2\). Use Archimedes’s Theorem to compute this area precisely.

Ans: 58.499, \( \frac{176}{3} \approx 58.667 \).

57. Consider the equation \( f(x) = \sqrt{4 - x^2} \). Its graph is the upper half of the circle of radius 2 with center the origin. Select the points

\[
0 \leq 0.2 \leq 0.4 \leq 0.5 \leq 0.6 \leq 0.8 \leq 1 \leq 1.2 \leq 1.3 \leq 1.5 \leq 1.6 \leq 1.8 \leq 1.9 \leq 2
\]
on the \( x \)-axis between 0 and 2, and compute the sum of the areas \( y \cdot dx = f(x) \cdot dx \) of all the rectangles that this setup determines (again with three decimal place accuracy). Observe that this sum is an estimate of the area under the upper half of the circle and over the segment from 0 to 2 on the \( x \)-axis. What is this area equal to precisely?

Ans: 3.268, \( \pi \approx 3.146 \).

### Computing Areas and Integrals by using the Fundamental Theorem

58. Sketch the graph of the function \( f(x) = \sqrt{x} + 2 \). Compute the area under the graph and over the \( x \)-axis from \( x = 0 \) to \( x = 9 \).

Ans: 36

59. Consider the function \( f(x) = x^3 + 1 \) for \( 0 \leq x \leq 4 \). Sketch its graph and compute the area under the graph and over the \( x \)-axis.

Ans: 68

60. Consider the parabolic section obtained by cutting the parabola \( y = -3x^2 + 2x + 1 \) with the \( x \)-axis. Express the area of the parabolic section as a definite integral. Compute this area by applying the Fundamental Theorem of Calculus and then again by using Archimedes’s Theorem.
61. Consider the parabolic section obtained by cutting the parabola $y = -x^2 + 7x - 6$ with the line $y = 2$. Express the area of the parabolic section as a definite integral. To compute the area consider the application of both the Fundamental Theorem of Calculus and Archimedes’s Theorem. Choose the simpler of the two methods.

Ans: $\frac{17}{6} \sqrt{17}$

62. Explain why $\int_a^b (f(x) + g(x)) \, dx$ is equal to $\int_a^b f(x) \, dx + \int_a^b g(x) \, dx$ first by appealing to the definition, and then again by using to the Fundamental Theorem of Calculus.

Using Derivatives and Integrals

63. Consider a polynomial of the from $f(x) = ax^2 + bx + c$ with $a > 0$ and suppose that it has two (real) roots. Locate the two roots on the $x$-axis and compute the midpoint between them. How is this point related to the minimum value of the function $f(x)$? What property of the graph of $f(x)$ confirms this connection.

Ans: $(\frac{-b}{2a}, 0)$, $f'(x) = 0$ for $x = \frac{-b}{2a}$.

64. Use derivatives to compute the distance between the line $y = -\frac{1}{2}x + 5$ and the point $(-4, 3)$. [Hint: Let $(x, y)$ be any point on the line. Express the distance between $(-4, 3)$ and $(x, y)$ as a function of $x$. Then determine the smallest value of the square of this distance.]

Ans: $\frac{8}{\sqrt{5}}$

65. Consider the parabola $y = \frac{1}{2}x^2$ and cut it with the line from $(-1, \frac{1}{2})$ to $(4, 8)$ to obtain the parabolic section shown above. Show that the equation of the line of the cut is $y = \frac{3}{2} + 2$.

i. Make use of the Fundamental Theorem to compute the area $A$ of the parabolic section.

ii. Compute the coordinates of the vertex $V$ of the parabolic section.

iii. Use calculus to compute the distance from $V$ to the line of the cut. [Hint: Use the strategy of Exercise 64.]
iv. Compute the area of the parabolic section again, this time with Archimedes’s Theorem.

Ans: i. \( \frac{5}{12} \), ii. \( \left( \frac{3}{2}, \frac{9}{8} \right) \), iii. \( \frac{25}{4\sqrt{13}} \)

66. Draw the graph of the function \( f(x) = \sqrt{x} \). Let \( Q \) be a point on the graph and take the point \( P \) on the \( x \)-axis so that the segment \(QP\) is perpendicular to the axis. Let \( A \) be the area of the region under the graph of \( f \) and over the segment from the origin \( O \) to \( P \). Let \( B \) be the area of the triangle determined by the tangent to the graph at \( Q \), the segment \( PQ \), and the \( x \)-axis. Show that \( A = \frac{2}{3}B \) no matter where \( Q \) is taken.

67. Consider the function \( f(x) = 2x^2 - x^4 = x^2(2 - x^2) \).

i. Show that \( f(x) = 0 \) for \( x = -\sqrt{2}, 0, \) and \( \sqrt{2} \), but for no other \( x \).

ii. Show that \( f(x) \geq 0 \) for \( -\sqrt{2} \leq x \leq \sqrt{2} \) and that \( f(x) < 0 \) for all other \( x \).

iii. Show that \( f'(x) = 4x(1 - x^2) \).

iv. What is the slope of the tangent to the graph at \( x = -\sqrt{2} \)? At \( x = \sqrt{2} \)? Find those \( x \) at which the graph has a horizontal tangent.

v. For what values of \( x \) does \( f \) achieve its largest value? What is the largest value of \( f \)?

68. Continue to consider the function \( f(x) = 2x^2 - x^4 = x^2(2 - x^2) \) and its derivative \( f'(x) = 4x(1 - x^2) \).

i. Start with a large negative \( x \) on the \( x \)-axis. Move towards \( x = -1 \). Is \( f'(x) \) increasing or decreasing in the process? Now start at \( x = 1 \) and move to the right. Is \( f'(x) \) increasing or decreasing? What do your answers tell you about the slopes of the tangent lines of the graph of \( f \)?

ii. Use all the information you have about \( f \) and \( f' \) to sketch the graph of \( f \).

iii. Apply the Fundamental Theorem to show that the area under the graph of \( f(x) \) from \( -\sqrt{2} \) to \( 0 \) is equal to the area under the graph of \( f(x) \) from \( 0 \) to \( \sqrt{2} \). What basic feature of the graph of \( f \) is this fact related to?

Definite Integrals and Lengths of Circular Arcs

69. Consider the circle \( x^2 + y^2 = 4 \). Refer to the diagrams below, and show that the \( x \)-coordinate

![Diagrams](image_url)

of the point \( B \) is 1 in (a), \( \sqrt{2} \) in (b), and \( \sqrt{3} \) in (c). It follows, as in the discussion of the
circle of radius 5 that concludes Section 5.2, that the slope of the tangent to the circle at any point \((x, y)\) with \(y \neq 0\) is \(-\frac{x}{y}\). Use this to show that the derivative of \(f(x) = \sqrt{4 - x^2}\) is \(f'(x) = \frac{-x}{\sqrt{4 - x^2}}\). Deduce from this and the diagrams that
\[
\int_0^1 \frac{1}{\sqrt{4 - x^2}} \, dx = \frac{\pi}{6}, \quad \int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}} \, dx = \frac{\pi}{4}, \quad \text{and} \quad \int_0^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}} \, dx = \frac{\pi}{3}.
\]

70. Consider the function \(f(x) = \sqrt{r^2 - x^2}\). The figure below shows its graph, the point \(A = (0, r)\), a point \(B\) in the first quadrant, and the angle \(\theta\) with \(0 \leq \theta \leq \frac{\pi}{2}\) that \(B\) determines. Use the fact that \(f'(x) = \frac{-x}{\sqrt{r^2 - x^2}}\), to verify the equality
\[
\int_0^{r \cos \theta} \frac{1}{\sqrt{r^2 - x^2}} \, dx = \frac{\pi}{2} - \theta.
\]

Definite Integrals as Areas, Volumes, and Lengths of Curves

71. The definite integral \(\int_1^5 \sqrt{1 + 4x^2} \, dx\) is both
i. the area under the graph of the function \(f(x) = \) ___________ over the interval \(__ \leq x \leq __\), and
ii. the length of the graph of the function \(f(x) = \) ___________ from the point (__, __) to the point (__, __).

72. The definite integral \(\int_0^3 \sqrt{1 + x} \, dx\) is
i. the area under the graph of \(f(x) = \) ___________ from \(x = __\) to \(x = __\), as well as
ii. the volume obtained by rotating a region under the graph of \(g(x) = \) ___________ one revolution about the \(x\)-axis, and also
iii. the length of a piece of the graph of \(h(x) = \) ___________.

73. Show that the graph of \(f(x) = \sqrt{x^2 + 4^2}\) is the upper half of a hyperbola and sketch it.

i. Compute the volume of the solid obtained by rotating the region below the graph and above the segment \(2 \leq x \leq 6\) one complete revolution around the \(x\)-axis.
ii. Use Leibniz’s tangent method for the curve \(y^2 = x^2 + 4^2\) to show that the derivative of
the function $f$ is $f'(x) = \frac{x}{\sqrt{x^2+4}}$.

iii. Express as a definite integral the length of the hyperbolic arc from the point $(0, 4)$ to the point $(3, 5)$.

iv. Compute the area under the graph of the function $g(x) = \frac{x}{\sqrt{x^2+4}}$ and above the segment $1 \leq x \leq 5$.

Ans: i. $400 \pi / 3$, iv. $\sqrt{41} - \sqrt{17}$

74. Express as definite integrals the volumes obtained by rotating the graphs of the functions $y = \sin x$, $0 \leq x \leq \frac{\pi}{2}$, and $y = \cos x$, $0 \leq x \leq \frac{\pi}{2}$, one revolution around the $x$-axis. Compute each of these volumes.

About Hyperbolas

75. Determine the $x$ and $y$ intercepts of the hyperbolas $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ and $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ and their asymptotes. Sketch their graphs.

76. Sketch the asymptotes of the hyperbolas $x^2 - \frac{y^2}{b^2} = 1$ for $b$ equal to $\frac{1}{2}$, 1, and 2 on the same axis system. Draw in the graphs of the three hyperbolas.

77. Sketch the asymptotes of the hyperbolas $\frac{x^2}{a^2} - y^2 = 1$ for $b$ equal to $\frac{1}{4}$, 1, and 4 on the same axis system. Draw in the graphs of the three hyperbolas.

78. Sketch the asymptotes of the hyperbolas $y^2 - \frac{x^2}{b^2} = 1$ for $b$ equal to $\frac{1}{2}$, 1, and 2 on the same axis system. Draw in the graphs of the three hyperbolas.

79. Study the solution of Exercise 26 of Chapter 4 and then show that the focal points of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $(-\sqrt{a^2+b^2}, 0)$ and $(\sqrt{a^2+b^2}, 0)$.

An Assertion of Leibniz

80. Leibniz asserts the following (see the bottom of the right column on page 125): “And it was not difficult for me to figure out that the description of this curve could be reduced to the quadrature of the hyperbola.” Refer to Exercise 19. The Fundamental Theorem of Calculus tells us that the problem of finding an antiderivative of $y = \frac{\sqrt{a^2-x^2}}{x}$ (the $-$ sign is not relevant in this regard) is closely related to the problem of finding the area under the graph of this function. So this is the “quadrature” that Leibniz appears to have had in mind. Thus Leibniz seems to think that the graph of $y = \frac{\sqrt{a^2-x^2}}{x}$, for $0 < x \leq a$, lies on a hyperbola. Is this correct? [Hint: Recall from Section 5.1 that any hyperbola (in fact any conic section) is given by an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ for some constants $A, B, C, D, E, and F$, not all of which are zero. Assuming that Leibniz is correct, $y = \frac{\sqrt{a^2-x^2}}{x}$ satisfies such an equation. This implies, after substituting and rearranging things, that
\[ C(a^2 - x^2) + E \sqrt{a^2 - x^2} = -Ax^4 - Bx^2 \sqrt{a^2 - x^2} - Dx^3 - Fx^2 \]

for all \( x \) with \( 0 < x \leq a \). Now push \( x \) to zero and conclude that \( C \) must be zero. After canceling \( x \), push \( x \) to zero again and conclude that \( E = 0 \). So \( B \sqrt{a^2 - x^2} = -Ax^2 - Dx - F \). Show that the graph of \( y = B \sqrt{a^2 - x^2} \) has a vertical tangent at \( x = a \) and conclude that \( B = 0 \). But this implies that \( Ax^2 + Dx + F = 0 \) for \( 0 < x \leq a \). So all the constant \( A, B, C, D, E, \) and \( F \) are zero.

### Applications of Pappus’s Theorems

81. Consider a line segment and an axis in the plane. The segment has length \( s \). One of the endpoints of the segment touches the axis (without crossing it) and the other endpoint of the segment is a distance \( r \) away from the axis. Notice that by rotating the segment one complete revolution around the axis we get the surface area (slanted part only) of a cone of radius \( r \) and slanted height \( s \). Use one of Pappus’s theorems to show that this surface area is \( rs\pi \).

82. Consider a rectangle with sides \( a \) and \( b \). Whether considering just the perimeter or the “solid” rectangle, the centroid in either case is the center \( C \) of the rectangle. Suppose that the distance from \( C \) to the axis is \( d \). Consider the solid obtained by rotating the rectangle one full revolution about the axis. Show that the surface area of this solid is \( 4\pi d(a + b) \) and that its volume is \( 2\pi dab \). Why are these answers the same for any axis of rotation in this plane as long as the distance from \( C \) to the axis is the same \( d \)? Rotate the rectangle with respect to an axis that is parallel to one of its sides. Then check the answers above by using the formulas for the surface area and the volume of a cylinder.

83. A hollow cylindrical pipe is 4 inches long, has an inner radius of 1 inch and is \( \frac{1}{4} \) inch thick.

Use one of Pappus’s theorems to compute the volume of the pipe (solid part only).
84. In the triangle below, \( h \) is the height relative to the base \( b \). Determine the distances from the base \( b \) of (i) the centroid \( C_1 \) of the triangle (just the boundary) and (ii) the centroid \( C_2 \) of the triangular region (boundary and interior) in terms of \( a, b, c, \) and \( h \).