Additional Exercises for Chapter 6

The Coefficients of the Product \((y + \Delta y)^n\)

The professor of a class of 38 students wishes to select five students to form a team that will collaborate in the analysis and solution of a calculus problem. In how many different ways can he select the five students? Whatever the number of different teams is, write this number by \((\binom{38}{5})\). Let \(S\) be a set that has \(n\) elements. Let \(i\) be any number less than or equal to \(n\), and let \(\binom{n}{i}\) be the total number of different subsets of \(S\) that have \(i\) elements. In the discussion below you will need to know that for any positive integer \(k\), the symbol \(k!\) stands for the product \(k \cdot (k-1) \cdot (k-2) \cdots 3 \cdot 2 \cdot 1\).

25. Experiment with the definition above. For example, what are \(\binom{5}{1}\), \(\binom{3}{3}\), and \(\binom{3}{2}\) equal to? Explain why \(\binom{5}{3} = \binom{4}{2} + \binom{4}{1}\). Finally explain why \(\binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i}\).

26. Show that \(\binom{5}{3} = \frac{5!}{3!2!}\). Use the principle of Mathematical Induction and the last equation in the previous exercise to show that \(\binom{n}{i} = \frac{n!}{i!(n-i)!}\). In how many different ways can the professor select the five students from the 38?

27. Study the way the product \((y + \Delta y)^n\) was expanded in Section 1. Use the same strategy to show that \((y + \Delta y)^3 = y^3 + 3y^2\Delta y + 3y(\Delta y)^2 + (\Delta y)^3\). Use this strategy once more to derive a formula \((y + \Delta y)^4\). [Hint: Use fact that there are \(\frac{4!}{2!2!} = 6\) different ways in which 2 elements can be chosen from a 4 element set.] Check your formula by multiplying things out “longhand”. Write down a formula for the product \((y + \Delta y)^6\).

Some of Newton’s Stuff

28. Study Newton’s proof of the fact that the derivative of \(f(x) = k \cdot x^m\) is \(f’(x) = k \cdot \frac{m}{n} \cdot x^{\frac{m}{n}-1}\).
   i. Provide the details of Newton’s argument in showing that the derivative of \(f(x) = x^{\frac{3}{2}}\) is \(f’(x) = \frac{3}{2} x^{\frac{1}{2}}\).
   ii. Do the same to show that the derivative of \(f(x) = x^{\frac{1}{4}}\) is \(f’(x) = \frac{1}{4} x^{-\frac{3}{4}}\).

29. Consider the function \(f(x) = \sqrt{x} + 2\). For any \(x \geq 4\), let \(A(x)\) be the area under the graph of \(f\) over the interval from 4 to \(x\). Use Newton’s approach to the Fundamental Theorem of Calculus to determine \(A(x)\) explicitly.
   Answ: \(\frac{2}{3} x^{\frac{3}{2}} + 2x - \frac{40}{3}\)

30. Consider the function \(g(x) = 1 + x + x^2\). For any \(x \geq 0\), let \(A(x)\) be the area under the graph of \(g\) over the interval from 0 to \(x\). Use Newton’s approach to the Fundamental Theorem of Calculus to determine \(A(x)\) explicitly.
   Answ: \(\frac{1}{3} x^3 + \frac{1}{2} x^2 + x\)

31. Consider the function \(h(x) = x^2 + x^{\frac{1}{4}}\). For any \(x \geq -8\), let \(A(x)\) be the area under the graph of \(h(x)\) over the interval from \(-8\) to \(x\). Use Newton’s approach to the Fundamental Theorem of Calculus to determine \(A(x)\) explicitly.
Answ: $\frac{1}{3}x^3 + \frac{3}{5}x^{5/3} + \frac{512}{3} + \frac{96}{5}$

32. The discussion of Newton’s proof of the Fundamental Theorem of Calculus in Section 6.2 assumes that the function $f$ satisfies $f(x) \geq 0$ for all $x$. Now let $f$ be a general function with a graph that is all in one piece from $x \geq a$ onward. Let $A(x) =$ sum of areas above the $x$-axis (and below that graph) minus the sum of areas below the $x$-axis (and above the graph) from $a$ to $x$. Show that $A(x)$ is an anti-derivative of $f(x)$ for $x \geq a$.

**Definite Integrals and the Fundamental Theorem**

33. Are the functions $f(x) = x^2 + 3x$, $F(y) = y^2 + 3y$, and $\phi(z) = z^2 + 3z$ identical? Why, or why not?

34. Show that $\int_0^4 \sqrt{x} \, dx = \frac{16}{3}$. What are $\int_0^4 \sqrt{t} \, dt$, $\int_0^4 \sqrt{u} \, du$, and $\int_0^4 \sqrt{z} \, dz$ equal to?

35. (Fill in the blanks.) The function $f(x) = \sqrt{x}$ is given. The definite integral $\int_0^4 \sqrt{x} \, dx$ is equal to a number that depends only on ___. So the rule $\int_0^x \sqrt{x} \, dx$ defines a function. The values that this function assign to 1, 4, and 100 are ___, ___, and ____.

36. Define the functions $F$, $G$, and $H$ by the rules $F(x) = \int_2^x \frac{1}{\sqrt{t}} \, dt$, $G(x) = \int_2^x \frac{1}{\sqrt{t}} \, dt$, and $H(t) = \int_2^t \frac{1}{\sqrt{t}} \, dt$. Evaluate each of these functions at 4. Evaluate each of them at any number $c \geq 2$. Explain why these functions are the same by interpreting them in terms of areas. Define the function $K(x) = \int_1^x \frac{1}{\sqrt{t}} \, dt$. Show that $F$ and $K$ differ by a constant $C$ and interpret $C$ as an area.

37. Consider the function defined by the rule $x \rightarrow \int_3^x (t^2 + 5) \, dt$. Evaluate it at $x = 5$. What is the problem with defining this function by $x \rightarrow \int_3^x (x^2 + 5) \, dx$?

38. Consider the functions $F(x) = \int_0^x (t^2 + 3t) \, dt$, $G(x) = \int_2^x \frac{1}{\sqrt{t}} \, dt$, and $K(x) = \int_3^x \sqrt{t^3 + 5} \, dt$. Determine the derivatives $F'(x)$, $G'(x)$, and $K'(x)$.

39. Use a definite integral to define a function that has $f(x) = \frac{1}{x}$ as its derivative.

40. Use a definite integral to define an antiderivative of $g(x) = \sqrt{2x^2 + 4}$.

41. Consider the upper half of the circle of radius 1 with center at the origin. For any $x$ with $-1 \leq x \leq 1$, let $G(x)$ be the area under this curve (and above the $x$-axis) from $-1$ to $x$. Express $G(x)$ in terms of a definite integral. What is the derivative $G(x)$ equal to?

**Computing with Power Series**

42. Recall that the approximation $\frac{1}{1+x} \approx 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ is valid for any $x$ with $|x| < 1$.

i. Show that $\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \cdots$. For what $x$ is this approximation valid?

ii. Show that $\frac{1}{1+x^2} \approx 1 - \frac{1}{3}x^2 + \frac{1}{3\cdot 5}x^4 - \frac{1}{3\cdot 5\cdot 7}x^6 + \frac{1}{3\cdot 5\cdot 7\cdot 9}x^8 - \frac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11}x^{10} + \cdots$. For what $x$ is this approximation valid?
43. Recall that the approximation \( \sqrt{1+x} \approx 1 + \left( \frac{1}{2} \right)x + \left( \frac{1}{2}\right)^2x^2 + \left( \frac{1}{2}\right)^3x^3 + \left( \frac{1}{2}\right)x^4 + \cdots \) is valid for any \( x \) with \( |x| \leq 1 \).

i. Show that \( \sqrt{1+3x^2} \approx 1 + \left( \frac{1}{2}\right)3x^2 + \left( \frac{1}{2}\right)^23^2x^4 + \left( \frac{1}{2}\right)^33^3x^6 + \cdots \). What is the fifth term in this approximation? For what \( x \) is the approximation valid?

ii. Show that \( \sqrt{1+\frac{1}{2}x^2} \approx 1 + \left( \frac{1}{2}\right)x^{-2} + \left( \frac{1}{2}\right)^2x^{-4} + \left( \frac{1}{2}\right)^3x^{-6} + \left( \frac{1}{2}\right)x^{-8} + \cdots \). For what \( x \) is this approximation valid?

44. There is a connection between the number \( \left( \frac{1}{2} \right)^i \) and the number \( \left( \frac{n}{i} \right) \) studied earlier. What is it?

To solve problems 45, 46, and 47, use one of the power series discussed above.

45. Compute the length of the parabola \( y = x^2 \) between the points \((0, 0)\) and \((\frac{1}{2}, \frac{1}{4})\) with an accuracy of two decimal places. Consider the graph of the parabola and assess the reasonableness of your answer. Answ: 0.57

46. Compute the length of the graph of \( f(x) = \frac{1}{x} \) from the point \((1, 1)\) to the point \((10, \frac{1}{10})\) with an accuracy of two decimal places. Consider the graph of the function and assess the reasonableness of your answer. Answ: 9.15

47. Compute the length of the graph of \( f(x) = x^2 \) from \((0, 0)\) to \((1, 1)\) with an accuracy of two decimal places. Assess the reasonableness of your answer.

Particles Moving on a Line

48. What is the average velocity of the moving particle in each of the situations below:

i. The particle of Exercise 10 between \( t = -2 \) and \( t = 1 \).
   Answ: 0

ii. The particle of Exercise 11 between \( t = 0 \) and \( t = 2 \).
   Answ: -25

iii. The particle of Exercise 12 between \( t = 1 \) and \( t = 5 \).
   Answ: \(-\frac{3}{5}\)

49. What is the average acceleration of the moving particle in each of the situations of Exercise 48.
   Answ: 4, -2, \( \frac{72}{100} \)

50. A point starts moving on the \( x \)-axis at time \( t = -5 \). Its position at any time \( t \geq -5 \) is given by \( p(t) = 3t^5 - 65t^3 + 540t + 3950 \). Determine the velocity function and the times at which the point comes to a stop. Consider the point’s motion to be produced by a force and discuss the action of this force.
   Answ: at \( t = -3, -2, 2, \) and 3. \( a(t) = 60t^3 - 390t = 30t(2t^2 - 13) \)

51. Verify the following assertion of Galileo. Consider a body that starts from rest at time \( t = 0 \)
and moves along a straight line with constant acceleration. Show that the velocity at any
time \( t \geq 0 \) is equal to twice the average velocity over the time interval from 0 to \( t \).

52. To begin his serve, a tennis player tosses a ball vertically upward. The ball leaves his hand
at a distance of 6 feet from the ground. At a distance of \( 9\frac{1}{2} \) feet from the ground, the ball
stops at the top of its flight and is smashed forward by the action of the racquet a split second
thereafter. With what initial velocity does the ball leave the player’s hand?

From the Popular Literature

53. Analyze for accuracy the thinking of a fictional “analyst” in Jonathan Fraken’s piece “The
Fall” that appeared in Harper’s in April 2000.

“Discounting the minimal effects of wind drag at low velocities something ‘plummeting’ (a
thing of value ‘plunging’ in free fall) experienced an acceleration due to gravity of 32 feet per
second squared, and, acceleration being the second-order derivative of distance, the analyst
could integrate once over the distance the object had fallen (roughly 30 feet) to calculate its
velocity (42 feet per second) as it passed the center of a window 8 feet tall, and, assuming a
6-foot long object, and also assuming for simplicity’s sake a constant velocity over the interval,
derive a figure of approximately four tenths of a second of full or partial visibility.”

54. Marilyn vos Savant in her column in the November 19, 2000 issue of Parade Magazine posed
the following problem.

“Say that two motorboats on opposite shores of a river start moving toward each other, but
at different speeds. When they pass each other the first time, they are 700 yards from the
shore line. They continue to the opposite shore, then turn around and start moving toward
each other again. When they pass a second time, they are 300 yards from the other shore
line. How wide is the river?” [Neglect factors such as current. Assume that the boats move
at constant speeds, and in particular that the boats bounce of the shore like billiard balls
without loss of speed.]

This question, already asked in an earlier column, had generated controversy because
readers - including “plenty of mathematicians and other professionals” - pointed out that
solution leads to a system of four equations in five unknowns, and that such systems could
not be solved. But Marilyn persisted in her claim that the answer is 1800 yards. Who is
right? [Note that this is not a calculus problem.]

Points Moving in the Plane

The context for Exercises 55 - 61 is provided by Figure 6.11 of the text. A unit of length and a unit
of time are given. A point is moving in the \( x\)-\( y \) plane. The \( x\)- and \( y\)-coordinates of its position are
functions \( x = x(t) \) and \( y = y(t) \) of time \( t \).

55. The point’s position is given by \( x(t) = t \) and \( y(t) = 1 - t \) at any time \( t \geq 0 \). Along what
straight line does it move?

56. In the two situations below: What is the velocity of the point in the \( x \)-direction at any time \( t \), in the \( y \)-direction, and overall? What is the equation of the curve in the variables \( x \) and \( y \) along which the point moves? Sketch the curve and describe the motion of the point. How do the two motions differ?

i. \( x(t) = t \) and \( y(t) = \frac{1}{t} \) for \( t \geq 1 \).

ii. \( x(t) = \sqrt{t} \) and \( y(t) = \frac{1}{\sqrt{t}} \) for \( t \geq 1 \).

57. Describe in each of the two cases below, the acceleration and velocity of the point at any time \( t \geq 0 \) as well as the path along which it travels.

i. \( x(t) = 3t + 4 \) and \( y(t) = 2t - 7 \).

ii. \( x(t) = 2t \) and \( y(t) = -t^2 \).

58. The point’s position is given by \( x(t) = \sqrt{t} \) and \( y(t) = \sqrt{1-t} \). The motion starts at time \( t = 0 \). At what time must it end? Show that the point travels along the circle of radius 1. Make a sketch of the motion.

59. Describe the motion of the points given by the following coordinate functions and starting at the indicated times:

i. \( x(t) = \cos t \), \( y(t) = \sin t \), and \( t \geq 0 \).

ii. \( x(t) = 5 \cos t \), \( y(t) = 3 \sin t \), and \( t \geq 0 \).

iii. \( x(t) = t \), \( y(t) = \sin t \), and \( t \geq 0 \).

iv. \( x(t) = t \cos t \), \( y(t) = t \sin t \), and \( t \geq 0 \). [Hint: Think about the difference between this problem and (i).]

v. \( x(t) = \frac{1}{2} \cos t \), \( y(t) = \frac{1}{t} \sin t \), and \( t \geq \frac{\pi}{100} \). [Hint: Again think about the difference between this problem and (i).]

Suppose that the curve along which the motion of a point takes place is the graph of a function \( y = f(x) \). In this case the \( x \)-coordinate \( x = x(t) \) of the motion determines the \( y \)-coordinate by \( y(t) = f(x(t)) \). In words, \( y(t) \) is obtained by replacing \( x \) by \( x(t) \) in \( f(x) \). Therefore, \( y = f(x) \) and \( x = x(t) \) together determine the motion of the point.

60. Describe and sketch the motion of a point moving on the graph of \( f(x) = x^2 \), in each of the situations \( x(t) = t \) and \( x(t) = \sin t \) where \( t \geq 0 \) in each case.

61. Describe and sketch the motion of a point moving on the graph of \( f(x) = \sqrt{x} \), in each of the situations \( x(t) = t \) and \( x(t) = \sin t \) where \( t \geq 0 \) in each case.
Tennis Anyone?

A tennis court consists of a flat horizontal surface. The “in play” region is bounded by two rectangles, one within the other. Both rectangles are 78 feet in length. The larger one, for doubles play, is 36 feet wide, and the smaller one, for singles play, is 27 feet wide. The net divides the rectangles into two equal halves of 39 feet by 36 feet for doubles and 39 feet by 27 feet for singles. The diagram below shows the configuration of the in play area of the court. The two sides of length 36 feet that are parallel to the net are called baselines. The lines perpendicular to the net are called side lines. The two lines within the court connecting the two side lines for singles play are parallel to the net at a distance of 21 feet from the net. They are called service lines. The top of the net is three feet off the ground at the center and rises on both sides to a maximum height of $3\frac{1}{2}$ feet at the sides of the court. The final match of a singles tournament is in progress.

62. Professor Neil Delaney - a tennis player of considerable skill - hits a soft lob in the direction of his opponent. The ball leaves his racket at an angle of $65^\circ$ with the horizontal and a velocity of 40 feet per second. It is 3 feet off the ground and directly over the service line at this instant. The horizontal component of its velocity is perpendicular to the net. Delaney’s opponent, “Monk” Malloy, at the net and in line with Delaney’s shot, dashes back and, at a distance of 13 feet from the net, leaps “Sampras style” for a return smash. Will the ball go over Monk’s head? Or will he - given his height and athletic ability - be able to soar high enough to return the ball?
Answ: Can Monk reach a ball that is 11.19 feet off the ground?

63. Monk presses the attack and strokes a hard ball that lands just inside the base line. Delaney racing back and drawn off court, lunges for a defensive lob. The ball, 6 feet behind the base line and 2 feet above the ground when it is struck, flies off the racket at an angle of $60^\circ$ with the horizontal and a velocity of 55 feet per second. Monk, at center court, watches despairingly as the ball floats high over his head. But will it land in play?
Answ: The ball lands 83 feet from where Delaney struck it.

64. A boy watches the match intently from behind the 24 foot wire mesh fence that surrounds the tennis complex. During a break in the action he decides to toss an old tennis ball over the fence. When the ball leaves his hand it is 4 feet off the ground and has a speed of 36 feet per second. Does it have a chance to clear the fence?
Answ: A vertical toss from the given height would reach a height of 24.25 feet at that speed.

But now back to the action. It is match point for Delaney ...

65. Delaney, responding to Monk’s shot, bends low to hit a difficult volley. The ball is 1 foot off the ground, 2 feet from the net, and 4 feet inside the sideline when it is struck. It leaves the racket at an angle of $45^\circ$ with the horizontal and with a velocity of 28 feet per second. The ball floats in the general direction of the center of the net, the horizontal component of its motion at an angle of $30^\circ$ with the plane of net. Monk charges in and dives for the return ...

Draw a careful diagram of the path of Delaney’s shot from a “bird’s eye” perspective above the court. Put in the relevant numerical data. Then turn to the central question: Which of the following three possibilities describes the state of the match?

i. Delaney’s volley goes into the net. Monk Malloy wins the point.

ii. Delaney’s volley clears the net but lands out of play. Monk Malloy wins the point.

iii. Delaney’s volley clears the net and lands in play. Monk’s diving return misses. Game, set, and match to Delaney.

More Moving Points

The context for Exercises 66 and 67 is provided by Figure 6.11 of the text. A unit of length and a unit of time are given.

66. A point starts its motion at time $t = 0$ from the origin. It has an initial velocity of 5 in the direction of the point $(1, 2)$. Determine the initial velocities in the $x$-direction and in the $y$-direction. The accelerations of the point in the $x$-direction and $y$-direction are given by $x''(t) = 0$ and $y''(t) = 3$, respectively. Find an equation of the curve in the variables $x$ and $y$ along which the point moves. How does the point move along this curve? [Hint: complete the square to get a sense of the shape of the curve.]

67. A point starts its motion at time $t = 0$ from the origin. Its accelerations in the $x$-direction and $y$-direction are the constants $x''(t) = a_1$ and $y''(t) = a_2$ respectively. Let the initial velocities in the $x$ and $y$ directions be $v_1$ and $v_2$ respectively. Verify the following:

i. $a_1y(t) - a_2x(t) = (a_1v_2 - a_2v_1)t$.

ii. If $a_1 = a_2 = 0$ then either the point remains fixed at the origin or it moves along a straight line.

iii. If either $a_1 \neq 0$ or $a_2 \neq 0$, and $a_1v_2 = a_2v_1$, then the point moves along a straight line.

iv. If either $a_1 \neq 0$ or $a_2 \neq 0$, and $a_1v_2 \neq a_2v_1$, then $x = x(t)$ and $y = y(t)$ satisfy an equation of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$, where either $A$ or $C$ is not zero. This implies by the discussion that concludes Section 5.1 that the point travels along a conic section.
v. Suppose $a_1 > 0$ and $a_2 > 0$. Consider the point to be driven by a force $F$. Draw on an $x$-$y$ coordinate system arrows that represent the components of $F$ in the $x$ and $y$ directions. Using the parallelogram law for forces, draw an arrow that represents $F$. Rotate the $x$-$y$ coordinate system (keeping the origin fixed) in such a way that $F$ points in the direction of the positive $x$-axis. Working with this rotated situation, show that the path of the point is a parabola.

vi. Conclude that the path is a parabola in the original situation as well.